Recovering the Variance Premium

Steven Heston
University of Maryland, College Park
sheston@rhsmith.umd.edu
November 2017
Preliminary

Ross (2015) Recovery:

- Recover “physical” probabilities from options.
- Limitations:
  – Requires a stationary state space.
  – Too good to be true (binomial or Black-Scholes).
- Relies on interest rate variation.
  – Constant interest rates just recover risk-neutrality.
    - Predicts that forward measure is risk-neutral.
    - Predicts long bond is log-efficient.
    - These predictions are obviously false.

New Generalized Recovery

- Heston’s (2004) *Path-Independence* extends Ross Recovery Theorem to power utility.
  - Log(pricing kernel) can be cointegrated with stock.
- New restrictions between equity premium and variance premium.
  - A long “power security” is log-efficient.
- Measure and test the variance premium.

Heston (1993) Model

- Risk-neutral dynamics:

  $dS = rSdt + \sqrt{v}Sdz_1^*,$ (1)

  $dv = \kappa^*(\theta^* - v)dt + \sigma\sqrt{v}Sdz_2^*.$

- Observable "physical" dynamics:

  $dS = (r+\mu v)Sdt + \sqrt{v}Sdz_1,$ (2)

  $dv = \kappa(\theta - v)dt + \sigma\sqrt{v}dz_2.$

- Martingale Condition:

  $U(t)M(t) = E_t[U(t+\Delta)M(t+\Delta)].$

What is *M(t)*?

- Proposition 1:

  Risk-neutral (1) and physical dynamics (2) imply a unique *M(t)*.

  $$M(t) = S(t)^{\gamma} exp(\beta t + \eta \int_0^t v(s)ds + \xi v(t)).$$

- Solve or invert the interest rate *r*, equity premium μ and variance premium $\kappa^* - \kappa$ in terms of β, γ, η, and ξ.
  - Impose economic restrictions.
  - I hate that path-dependent η term!

Merton's (1973) Bucket Shop Assumption

- Bucket Shop Assumption on option value:

  $U(t) = U(S(t),v(t),t).$

- Ross's Transition-Independence Assumption:

  $M(t) = M(S(t),v(t),t) = e^{\beta t}h(S(t),v(t)).$

  Price kernel should depend on where we are, not how we got there (through diffusion, jumps, etc.).

- $M(t)$ should not depend on $\int_0^t v(s)ds$.
  - The state space $\{S(t),v(t)\}$ should be enough.
  - Habit persistence could be incorporated into current state variables.

Path-Independence

- Constant rate of time preference β.
- *M* should be homogeneous in *S(t)*.
  - Returns do not depend on level of *S(t)*.
  - Options depend on moneyness, not level of *S(t)*.

- $M(t) = e^{\beta t} S(t)^{\gamma} h(v(t)),$

  where reciprocal marginal utility $N(v) = 1/h(v)$ satisfies the P.D.E. of Linetsky and Qin (2016):

  $½\sigma^2 v N'' + [\kappa^*(\theta^* - v) - \rho\sigma\gamma v]N' + ½\gamma(\gamma+1)v - \beta - (\gamma+1)r]N = 0.$

Recovery Theorem

- Given γ and risk-neutral dynamics (1), Proposition 1 shows all path-independent pricing kernels that give stationary physical dynamics.

  $h(v) = e^{\xi v(t)}$,

  where $\xi > 0$ satisfies a quadratic equation to make $\eta = 0$.

  If $\gamma < 0$, then there is only one positive root.
- We have recovered the physical dynamics (2).
  - Does not recover the mean in Black-Scholes unless you know γ.
- This works in more general models.

Valuation of a "Power" Security

- P.D.E.:

$$\tfrac{1}{2}vS^2U_{SS} + \rho\sigma vSU_{Sv} + \tfrac{1}{2}\sigma^2 vU_{SS}$$

$$+ rSU_S + \kappa^*(\theta^* - v)U_v - rU + U_t = 0.$$

- Terminal Payoff:

$$U(S,v,t;\phi,T) = S(T)^{\phi}.$$

- Solution:

$$U(S,v,t;\phi,T) = S(t)^{\phi}e^{C(T-t)+D(T-t)v(t)},$$

where $C(.)$ and $D(.)$ are complicated.

Long-Term Power Security

- When $\phi = -\gamma$, $D(\infty) \rightarrow -\xi$.
  - Long-term option prices reveal variance preference!

$$U(S(t), v(t), t; -\gamma, T) =$$

$$E_t\left[\frac{U(S(T),v(T),T;-\gamma,T)\,M(S(T),v(T),T)}{M(S(t),v(t),t)}\right].$$

Model-Free Test

- When $\phi = -\gamma$, the gross return on a long Power Security is the reciprocal of marginal rate of substitution.

$$R_\infty(t+\Delta) \equiv \lim_{T\to\infty} \frac{U(S(t+\Delta), v(t+\Delta), t+\Delta; -\gamma, T)}{U(S(t), v(t), t; -\gamma, T)} = \frac{M(S(t), v(t), t)}{M(S(t+\Delta), v(t+\Delta), t+\Delta)}.$$

- i.e., the long-term Power Security is growth optimal.
- Use Breeden-Litzenberger to construct power security from vanilla options.
- This even works when the Power Security uses a proxy S^*, as long as $\log(S^*)$ is cointegrated with $\log(S)$, which is cointegrated with $\log(M)$.

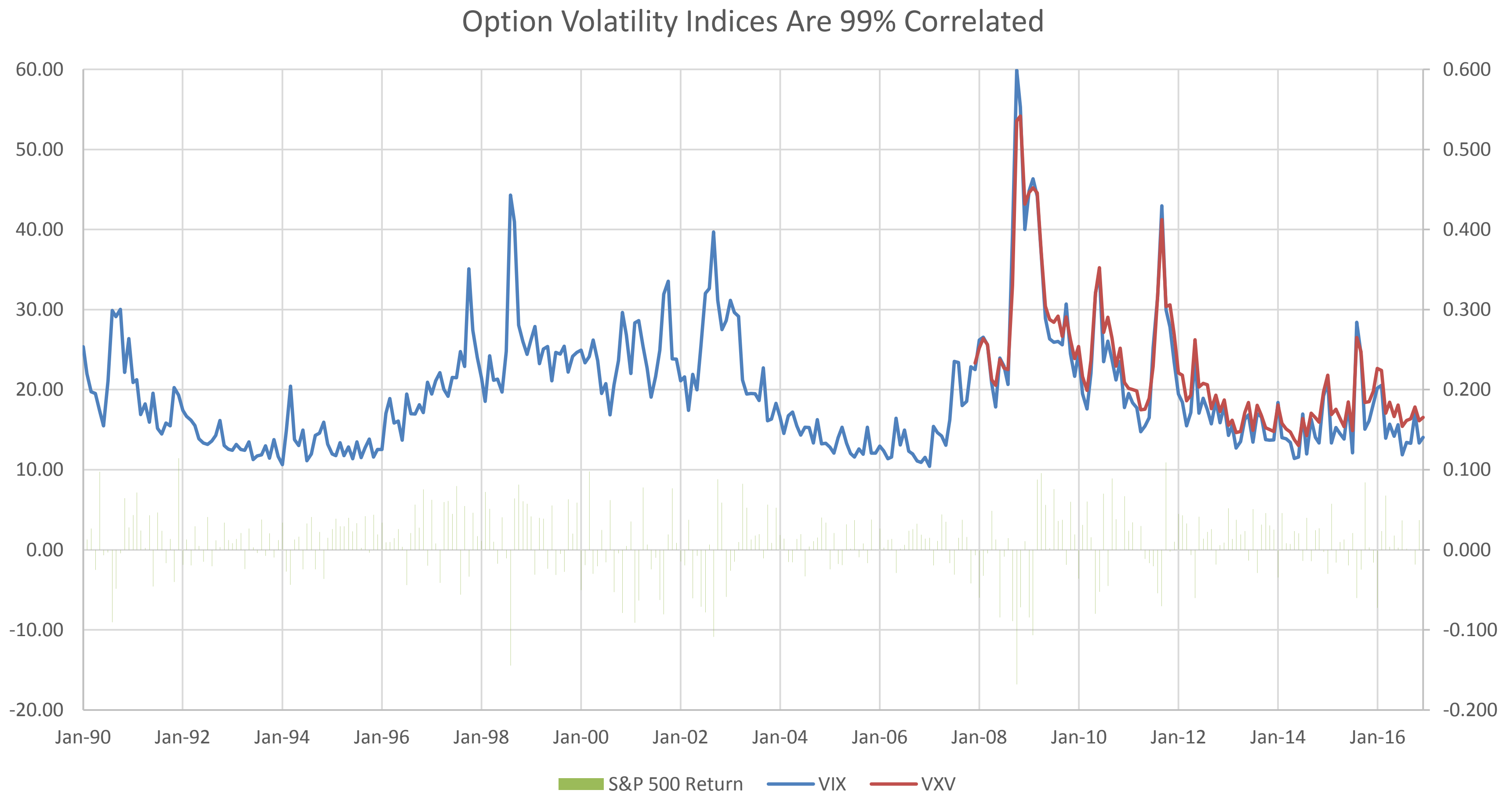
Estimating the Variance Premium

- If $\gamma, \rho < 0$, then the model predicts a positive equity premium and negative variance premium.
- Two strategies (which nobody reports!):
  - Monthly VIX^2 portfolio.
    - Adjust for exact number of days in trading month.
    - Portfolio has log-payoff,
    - 99% correlated with variance swap.
  - Bimonthly VXV^2 portfolio.
    - Buy the 3-month VXV^2 portfolio,
    - sell 2 months later using the VIX^2 price.
    - 99% correlated with two-month variance swap + VIX^2.

Monthly Data

- CRSP risk-free T-bill return.
- CBOE *S&P 500* Total Return Index.
- *VIX* 1990-2016 (27 years).
- *VXV* 2008-20016 (only 9 years).

VIX and *VXV*


Option Volatility Indices Are 99% Correlated
60.00
50.00
40.00
30.00
20.00
10.00
0.00
-10.00
-20.00
0.600
0.500
0.400
0.300
0.200
0.100
0.000
-0.100
-0.200
Jan-90
Jan-92
Jan-94
Jan-96
Jan-98
Jan-00
Jan-02
Jan-04
Jan-06
Jan-08
Jan-10
Jan-12
Jan-14
Jan-16
S&P 500 Return
VIX
VXV

Monthly Summary Statistics, 1990-2016 (VXV is 2008-2016)

Summary Statistics of Monthly Data, 1990-2016								
						Correlations		
		Standard	Auto-		Risk-Free	S&P500	VIX	VXV
	Mean	Deviation	Correlation		Return	Return	Return	Return
VIX	19.8	7.5	0.84					
VXV	23.0	8.3	0.87					
Risk-free return	0.2%	0.2%	0.98		1.00	0.03	0.08	0.27
S&P return	0.9%	4.2%	0.04		0.03	1.00	-0.19	-0.58
VIX return	-53.8%	59.4%	-0.09		0.08	-0.19	1.00	0.86
Bimonthly VXV return	-33.6%	85.4%	-0.05		0.27	-0.58	0.86	1.00

GMM Restrictions on Gross Returns $R_i(t)$ and Excess Returns

- Average (unconditional) equity premium:

  $E[(R_{S\&P}(t)-R_f(t))M(t)] = 0.$

- Average variance premium:

  $E[(R_{VIX}(t)-R_f(t))M(t)] = 0.$

- Average risk-free return (gives β):

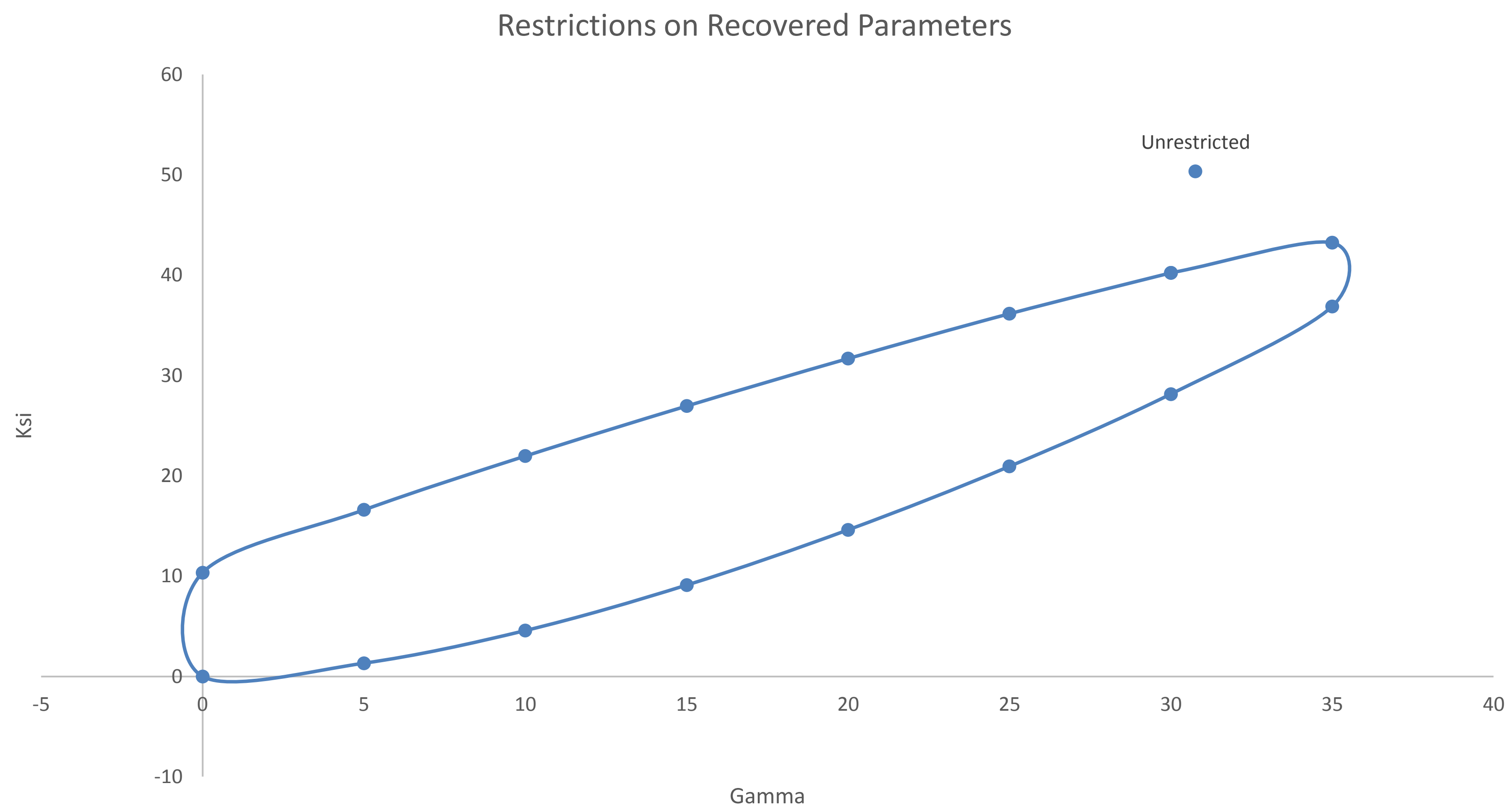
  $E[R_f(t)M(t)] = 1.$

- Conditional risk-free return:

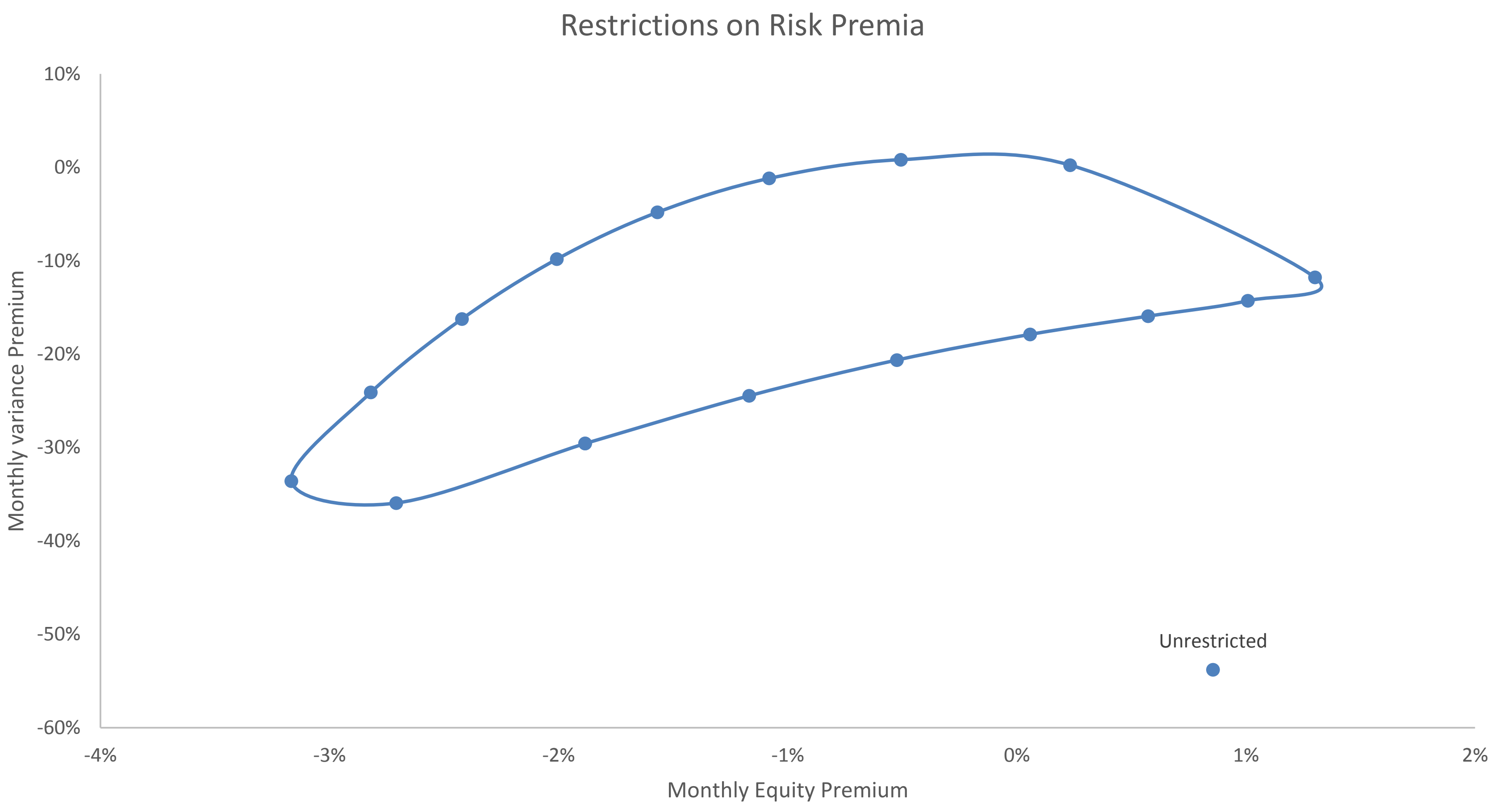
  $E[VIX^2(t)(R_f(t+\Delta)M(t+\Delta)-1)] = 0,$ or

  $Cov[VIX^2(t),R_f(t+\Delta)M(t+\Delta)] = 0.$

Recovery Restrictions



Restrictions on Risk Premia



Conclusion

- The pricing kernel $M(t)$ should jointly explain the cross-section of returns and the conditional predicted level.
- GMM does not reject with three parameters (β, γ, ξ) and four moments:
  - Unconditional equity premium,
  - Unconditional variance premium,
  - Unconditional risk-free return level,
  - Covariance between $VIX^2(t)$ and $R_f(t+1)$.