# A Discussion of: "The Shape of the Pricing Kernel and Expected Option Returns" by

**Christian Schlag**

Goethe University Frankfurt

**Tobias Sichert**

Stockholm School of Economics

**Discussant : Fousseni Chabi – Yo**

Umass Amherst

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# This Paper

- Use a parametric approach to provide a strong evidence that the projection of the Pricing Kernel (PK) on a single stock is U-shaped.
- A U-shaped PK has implications for the impact of volatility on expected option returns (an increase in volatility can lead to either increasing or decreasing expected call returns, depending on moneyness).
- A U-shaped PK offers a risk-based explanation for a number of option return patterns considered anomalous in the literature.

# Outline

Paper's Contribution

Comment 1

Comment 2

Comment 3

# The Pricing Kernel (PK): The Case of the Market Index

1. In a (one-period) representative agent economy, denote by $u[.]$ the investor's utility and assume that it is concave $(u^{''}[.] < 0)$. The PK $m_{t\to T}$ is

$$m_{t\to T} = \lambda_t u^{'}[W_t R_{M,t\to T}]$$

   where $\lambda_t > 0$

2. Since $u^{''}[.] < 0$, economy theory suggests that $m_{t\to T}$ decreases with the market return $R_{M,t\to T}$.

# U-Shaped PK: The Case of the Market Index

- Empirical evidence in Jackwerth (2000) suggests that the PK is U-Shaped (inconsistent with $u''[.] < 0$)
- Several explanations put forward to explain the PK puzzle (U-Shaped PK):
  - Chabi-Yo (2012) provides a market volatility explanation of the PK puzzle in a simple two-period (3-dates) economy without specifying the functional form of the utility: PK is a function of market return and market volatility.
  - Christoffersen et al. (2013) provide a market volatility explanation by using a parametric PK as a function of the market return and market volatility.

# U-Shaped PK and Expected Option Return

Bakshi et al. (2010):

1. Assume that the PK is U-Shaped
2. there exists a strike $K^*$ such that for $K > K^*$, the expected returns on call, digital calls, and kernel call (truncated PK) decreases with moneyness.
3. there exists a strike $\widetilde{K}$ such that for $K > \widetilde{K}$, the expected return on call, digital calls, and kernel call is negative.

Important:

- 1$\Longrightarrow$ 2 and 3
- 2 and 3 $\not\Rightarrow$1

# U-Shaped PK and Expected Option Return

- This paper relies on Bakshi et al. (2010), focuses on individual stocks and shows empirically that:
  - there exists a strike $K^*$ such that for $K > K^*$, the expected returns on call, and digital calls decreases with moneyness.
  - there exists a strike $\widetilde{K}$ such that for $K > \widetilde{K}$ the expected return on call and digital calls is negative.
- This paper concludes that the projected PK (PPK) on individual stocks is U-Shaped!
  - This is different from the statement in Bakshi et al. (2010)
  - Need a theoretical motivation: Why the pattern of expected call returns in terms of moneyness implies a U-Shaped PPK!

# Example with Unknown Utility in a One-Period Economy

1. The one-period PK $(m_{t\to T})$ :

$$\frac{\mathbb{E}_t m_{t\to T}}{m_{t\to T}} = \frac{\left(1/u'\left[W_t R_{M,t\to T}\right]\right)}{\mathbb{E}_t^*\left(1/u'\left[W_t R_{M,t\to T}\right]\right)} \text{ with } R_{M,t\to T} = \frac{S_T}{S_t}$$

2. Taylor expansion-series of PK around $R_{M,t\to T} = R_{f,t\to T}$ :

$$\frac{\mathbb{E}_t m_{t\to T}}{m_{t\to T}} = \frac{\sum_{i=0}^{3} a_i \left(R_{M,t\to T} - R_{f,t\to T}\right)^i}{\left\{1 + a_2 \mathbb{M}_t^{*(2)} + a_3 \mathbb{M}_t^{*(3)}\right\}}$$

with $a_0 = 1$, and $\mathbb{M}^{*(i)} = \mathbb{E}_t^*\left(R_{M,t\to T} - R_{f,t\to T}\right)^i$

$$\begin{aligned} a_1 &= \frac{1}{\tau R_{f,t\to T}} \quad \left(\frac{1}{\tau} = \text{relative risk aversion}\right) \\ a_2 &= \frac{(1-\rho)}{\tau^2 R_{f,t\to T}^2} \quad (\rho = \text{skewness preference}) \\ a_3 &= \frac{(1-2\rho+\kappa)}{\tau^3 R_{f,t\to T}^3} \quad (\kappa = \text{kurtosis preference}) \end{aligned}$$

# Example with Unknown Utility in a One-Period Economy

## Result 1: Conditional Expected Return

1. Call with maturity $T$ and strike $K_0^c$:

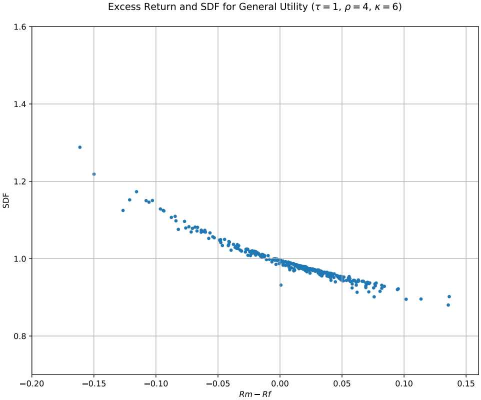
$$\underbrace{\mathbb{E}_t\left\{\frac{(S_T - K_0^c)^+}{C_t\left[K_0^c\right]}\right\}}_{\text{Expected Return on a Call}} = \sum_{i=0}^{3} a_i \frac{\mathbb{E}_t^*\left\{\left(R_{M,t\to T} - R_{f,t\to T}\right)^i (S_T - K_0^c)^+\right\}}{\left\{1 + a_2 \mathbb{M}_t{}^{*(2)} + a_3 \mathbb{M}_t{}^{*(3)}\right\} C_t\left[K_0^c\right]}$$

2. Digital Call with maturity $T$ and strike $K_0^c$:

$$\underbrace{\mathbb{E}_t\left\{\frac{1_{S_T > K_0^c}}{\frac{1}{R_{f,t\to T}}\mathbb{E}_t^*\left(1_{S_T > K_0^c}\right)}\right\}}_{\text{Expected Return on a Digital Call}} = \sum_{i=0}^{3} a_i \frac{\mathbb{E}_t^*\left\{\left(R_{M,t\to T} - R_{f,t\to T}\right)^i 1_{S_T > K_0^c}\right\}}{\left\{1 + a_2 \mathbb{M}_t{}^{*(2)} + a_3 \mathbb{M}_t{}^{*(3)}\right\} \mathbb{E}_t^*\left(\frac{1}{R_{f,t\to T}} 1_{S_T > K_0^c}\right)}$$

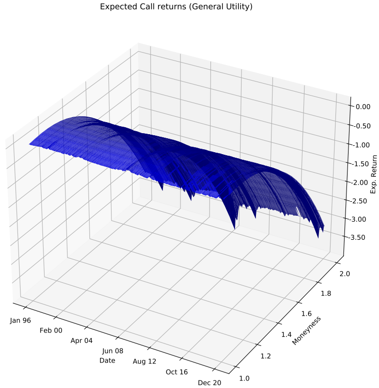
# Pricing Kernel ($\tau = 1, \rho = 4, \kappa = 6$)

SDF (x axis: realized excess market return)

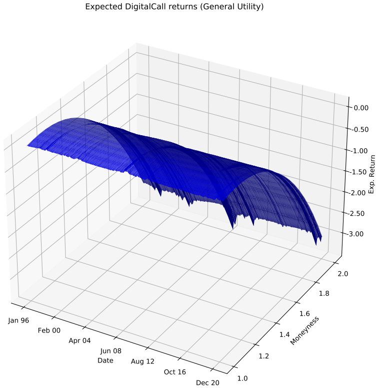


# Expected Option Return ($\tau = 1, \rho = 4, \kappa = 6$)

## Conditional (Call)

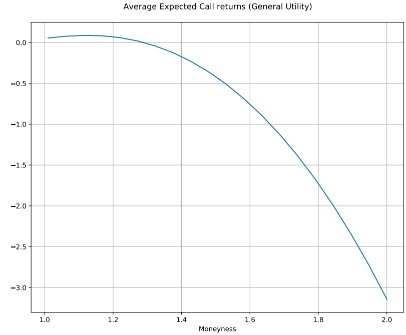



## Conditional (Digital Call)

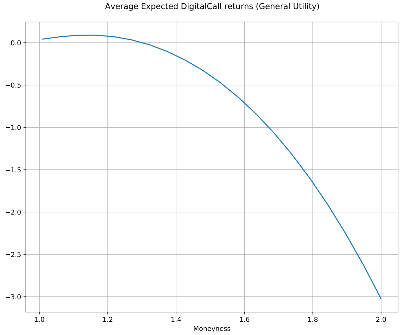


# Example ($\tau = 1, \rho = 4, \kappa = 6$)

## Unconditional (Call)

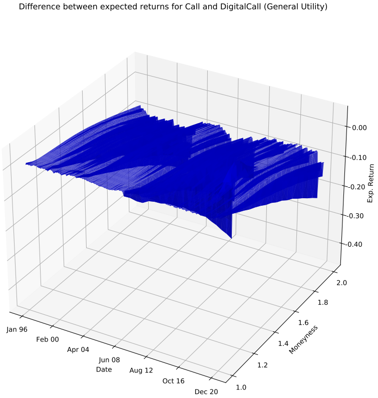



## Unconditional (Digital Call)

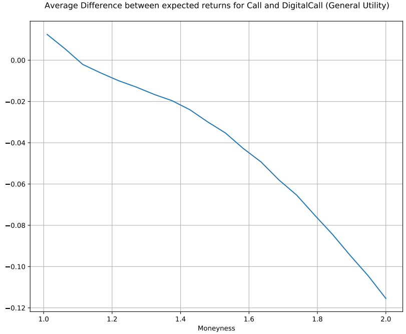


# Difference in Expected Option Returns $(\tau = 1, \rho = 4, \kappa = 6)$

Conditional (Call minus Dig Call)




Unconditional (Call minus Dig Call)



# Example with Unknown Utility in a One-Period Economy

1. Results are similar with alternative preference parameters $(\tau = 0.5, \rho = 4, \kappa = 8)$
2. Take away from this example: The patterns of expected return on call and digital call options do not imply a U-Shaped PK

# Outline

Paper's Contribution

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# Economic Theory Prediction about the Projected PK

1. The PK is:

$$m_{t\to T} = \lambda_t u^{'}\left[W_t R_{M,t\to T}\right]$$

Assume that: $u^{''}[.] < 0$ and $u^{'''}[.] > 0$. Denote the PPK as

$$m_{i,t\to T} = \lambda_t \mathbb{E}_t\left[u^{'}\left[W_t R_{M,t\to T}\right] | R^{(i)}_{p,t\to T}\right]$$

where $R^{(i)}_{p,t\to T} = (1-\omega_i) R_{f,t\to T} + \omega_i R_{i,t\to T}$.

2. First-Order Taylor expansion-series of $u^{'}$ around $\mathcal{R}_{M,t\to T} \approx \sum_{\substack{k=1\\k\neq i}}^{n} \omega_k R_{f,t\to T}$ allows to write the projected PK as

$$m_{i,t\to T} \simeq \lambda_t u^{'}\left[W_t R^{(i)}_{p,t\to T}\right] + \lambda_t \frac{W_t}{1!} u^{''}\left[W_t R^{(i)}_{p,t\to T}\right] \mathbb{E}_t\left[\mathcal{R}^{ex}_{M,t\to T} | R^{(i)}_{p,t\to T}\right]$$

where

$$\mathcal{R}^{ex}_{M,t\to T} = \sum_{\substack{k=1\\k\neq i}}^{n} \omega_k \left(R_{k,t\to T} - R_{f,t\to T}\right)$$

# Economic Theory Prediction about the Projected PK

1. Consider:

$$\mathbb{E}_t\left[\mathcal{R}^{ex}_{M,t\to T} | R^{(i)}_{p,t\to T}\right] = \widetilde{\beta}_{i,t} R^{(i)}_{p,t\to T} \text{ (assume } \alpha_i = 0 \text{ for simplicity)}$$

We have

$$\widetilde{\beta}_{i,t} = \beta_{i,t} - 1 \text{ with } \beta_{i,t} = \frac{\mathbb{COV}_t\left(R^{(i)}_{p,t\to T}, R_{M,t\to T}\right)}{\mathbb{VAR}_t\left[R^{(i)}_{p,t\to T}\right]}$$

and the derivative of the Projected PK (PPK) is:

$$\frac{\partial m_{i,t\to T}}{\partial R^{(i)}_{p,t\to T}} \simeq \lambda_t W_t u'' \left[W_t R^{(i)}_{p,t\to T}\right]\left(1 + \widetilde{\beta}_{i,t}\right) + \lambda_t W_t^2 u''' \left[W_t R^{(i)}_{p,t\to T}\right] \widetilde{\beta}_{i,t} R^{(i)}_{p,t\to T}$$

# Economic Theory Prediction about the Projected PK

1. Decreasing PPK Condition:

$$\frac{\partial m_{i,t\to T}}{\partial R^{(i)}_{p,t\to T}} < 0 \Leftrightarrow \beta_{i,t} > (\beta_{i,t} - 1)\,\rho_i\left[W_t R^{(i)}_{p,t\to T}\right]$$

where

$$\rho_i\left[W_t R^{(i)}_{p,t\to T}\right] = \underbrace{-\frac{W_t R^{(i)}_{p,t\to T}\, u'''\left[W_t R^{(i)}_{p,t\to T}\right]}{u''\left[W_t R^{(i)}_{p,t\to T}\right]}}_{\text{Relative Prudence Function}}$$

2. Assume that $\beta_{i,t} > 1$, then

$$\frac{\partial m_{i,t\to T}}{\partial R^{(i)}_{p,t\to T}} < 0 \Leftrightarrow \frac{\beta_{i,t}}{(\beta_{i,t} - 1)} < \rho_i\left[W_t R^{(i)}_{p,t\to T}\right]$$

3. Assume that $\beta_{i,t} < 1$, then

$$\frac{\partial m_{i,t\to T}}{\partial R^{(i)}_{p,t\to T}} < 0 \Leftrightarrow \frac{\beta_{i,t}}{(\beta_{i,t} - 1)} > \rho_i\left[W_t R^{(i)}_{p,t\to T}\right]$$

# Economic Theory Prediction about the Projected PK

1. CRRA utility: $u[x] = \frac{x^{1-\alpha}}{1-\alpha}$, the PK is perfectly decreasing in the market index, $\rho_i \left[ W_t R^{(i)}_{p,t\to T} \right] = 1 + \alpha$
   1.1 Assume that $\beta_{i,t} > 1$, then: $\frac{\partial m_{i,t\to T}}{\partial R^{(i)}_{p,t\to T}} < 0 \Leftrightarrow \frac{\beta_{i,t}}{\left(\beta_{i,t} - 1\right)} < 1 + \alpha$
   1.2 Assume that $\beta_{i,t} < 1$, then $\frac{\partial m_{i,t\to T}}{\partial R^{(i)}_{p,t\to T}} < 0 \Leftrightarrow \frac{\beta_{i,t}}{\left(\beta_{i,t} - 1\right)} > 1 + \alpha$.
2. Exponential utility $u[x] = 1 - e^{-\alpha x}$, the PK is perfectly decreasing in the market index, $\rho_i \left[ W_t R^{(i)}_{p,t\to T} \right] = \alpha W_t R^{(i)}_{p,t\to T}$
   2.1 Assume that $\beta_{i,t} > 1$, then $\frac{\partial m_{i,t\to T}}{\partial R^{(i)}_{p,t\to T}} < 0 \Leftrightarrow \frac{\beta_{i,t}}{\left(\beta_{i,t} - 1\right)} < \alpha W_t R^{(i)}_{p,t\to T}$
   2.2 Assume that $\beta_{i,t} < 1$, then $\frac{\partial m_{i,t\to T}}{\partial R^{(i)}_{p,t\to T}} < 0 \Leftrightarrow \frac{\beta_{i,t}}{\left(\beta_{i,t} - 1\right)} > \alpha W_t R^{(i)}_{p,t\to T}$

# Economic Theory Prediction about the Projected PK

1. Economic theory predicts that the Projected PK on the market is a decreasing function of the market index
2. Economic theory does not predict whether the PPK is a decreasing (increasing) function of a single stock: In fact the Projected PK can have any shape.
3. Suggestions: Better motivate the theory in this paper.

# Outline

# A U-Shaped PK : Risk-Based Explanation of Anomalies

1 The PK in CHJ (I denotes the market index)

$$\frac{M_t}{M_0} = \left(\frac{S_{t,I}}{S_{0,I}}\right)^{\phi} \exp\left\{\delta t + \eta \sum_{s=1}^{t} h_{z,s} + \xi\left(h_{z,t+1} - h_{z,1}\right)\right\}$$

2 This paper uses a parametric stock specific PK (i denotes individual stocks)

$$\frac{M_{t,i}}{M_{0,i}} = \left(\frac{S_{t,i}}{S_{0,i}}\right)^{\phi_i} \exp\left\{\delta_i t + \eta_i \sum_{s=1}^{t} h_{z,s} + \xi\left(h_{z,t+1} - h_{z,1}\right)\right\}$$

Parameters $\phi_i$, $\eta_i$ and $\delta_i$ identified with the pricing conditions:

$$E_{t-1}\left[\frac{M_{t,i}}{M_{t-1,i}}\right] = \exp\left(-r\right) \text{ and } E_{t-1}\left[\frac{M_{t,i}}{M_{t-1,i}}\frac{S_{t,i}}{S_{t-1,i}}\right] = 1$$

# A U-Shaped PK: Risk-Based Explanation of Anomalies

1. A stock specific PK is not a projection of CHJ's PK on a single stock. How do we rationalize this parametric projected PK in a simple economic model?
2. Parameters of the PPK are choosen to explain return on individual stock and the safe asset: No guaranty that this PPK fits observed option prices well! How about using the restriction $E_{t-1}\left[\frac{M_{t,i}}{M_{t-1,i}}\left(S_{T,i}-K_i\right)^+\right]=C_t\left[K_i\right]$?
3. The pricing restriction on the risk-free asset is the same for all stock $i$. How is this used in the simulations?

# A U-Shaped PK: Risk-Based Explanation of Anomalies?

The authors argue that the results in Boyer and Vorkink (2014) and Byun and Kim (2016) are perfectly in line with a U-shaped PK and thus with a risk-based explanation

1. Replicate Boyer and Vorkink (2014) in each moneyness bin: there is no negative relation between the lottery-like characteristics of options and subsequent option returns.
2. Replicate Bali et al. (2011) in each moneyness bin. High MAX-Low MAX average return is negative for low moneyness and becomes positive as the moneyness increases.

# A U-Shaped PK: Risk-Based Explanation of Anomalies?

1. Not clear why these results have a risk-based explanation.
2. A risk-based story: We need a model that shows economic mechanisms that explain why Byun and Kim (2016) and Boyer and Vorkink (2014) return series are priced in the cross-section of call option returns.

# Conclusion

1. Great Paper !
2. Strong Empirical Results!
3. Need a better motivation on the theory part!
4. Carefull interpretation of the empirics is needed!

# Example with Unknown Utility in a One-Period Economy

## Result 2: Conditional Expected Return

1. Put with maturity $T$ and strike $K_0^p$

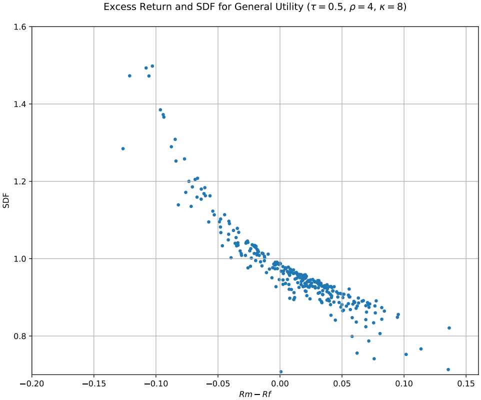
$$\underbrace{\mathbb{E}_t\left\{\frac{(K_0^p - S_T)^+}{P_t\left[K_0^p\right]}\right\}}_{\text{Expected Return on a Put}} = \sum_{i=0}^{3} a_i \frac{\mathbb{E}_t^*\left\{(R_{M,t\to T} - R_{f,t\to T})^i\,(K_0^p - S_T)^+\right\}}{\left\{1 + a_2\mathbb{M}^{*(2)} + a_3\mathbb{M}^{*(3)}\right\} P_t\left[K_0^p\right]}$$

2. Digital put with maturity $T$ and strike $K_0^p$

$$\underbrace{\mathbb{E}_t\left\{\frac{1_{K_0^p > S_T}}{\frac{1}{R_{f,t\to T}}\mathbb{E}_t^*\left(1_{K_0^p > S_T}\right)}\right\}}_{\text{Expected Return on a Digital Put}} = \frac{\sum_{i=0}^{3} a_i \mathbb{E}_t^*\left((R_{M,t\to T} - R_{f,t\to T})^i\, 1_{K_0^p > S_T}\right)}{\left\{1 + a_2\mathbb{M}^{*(2)} + a_3\mathbb{M}^{*(3)}\right\}\mathbb{E}_t^*\left(\frac{1}{R_{f,t\to T}} 1_{K_0^p > S_T}\right)}$$

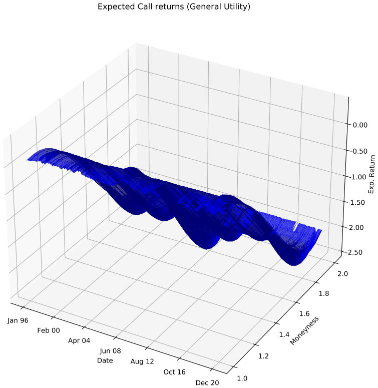
# Pricing Kernel (PK) ($\tau = 0.5, \rho = 4, \kappa = 8$)

## SDF (x axis: realized excess market return)

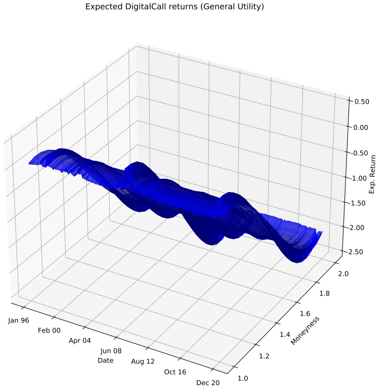


# Example ($\tau = 0.5, \rho = 4, \kappa = 8$)

## Conditional (Call)

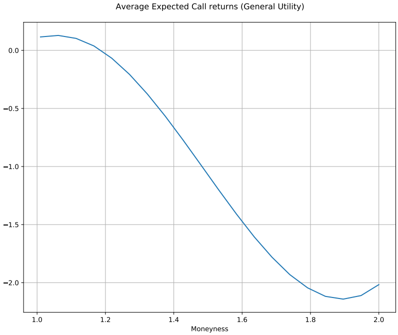



## Conditional (Digital Call)

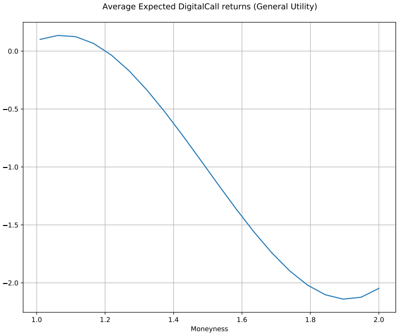


# Example ($\tau = 0.5, \rho = 4, \kappa = 8$)

## Unconditional (Call)

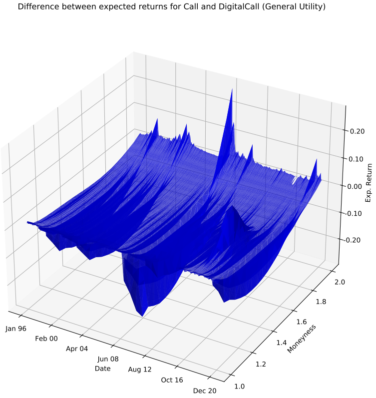



## Unconditional (Digital Call)



# Difference in Expected Option Returns $(\tau = 0.5, \rho = 4, \kappa = 8)$

## Conditional (Call minus Dig Call)




## Unconditional (Call minus Dig Call)