A Decomposition of Conditional Risk Premia and Implications for Representative Agent Models

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Abstract

We develop a methodology to decompose the conditional market risk premium and risk premia on higher-order moments of excess market returns into components related to contingent claims on down, up, and moderate market returns. The decompositions do not depend on assumptions about investor preferences, nor do they depend on assumptions about the market return distribution. Analogous decompositions implied by prominent representative agent models fail to match those implied by the data. Our results provide a host of new empirical facts regarding sources of conditional risk premia and identify a set of new challenges for representative agent models.

JEL Classification: E44; G1; G12; G13.

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Introduction

Identifying and understanding the sources of risk that generate the market risk premium is a fundamental challenge in financial economics. Well-known representative agent asset pricing models are able to generate the market risk premium using vastly different economic mechanisms including habit formation (e.g., Campbell and Cochrane, 1999; Bekaert and Engstrom, 2017), long-run risk (e.g., Bansal and Yaron, 2004; Drechsler and Yaron, 2011), and disaster risk (e.g., Barro, 2006; Gabaix, 2012; Wachter, 2013). Unconditionally, the market risk premia from such models match average excess market returns in the data, but, conditionally, each has very different implications for the sources of the risk premia. Extant literature lacks methodologies to estimate sources of conditional risk premia implied by the data and implied by models, making it difficult to evaluate the plausibility of disparate modeling assumptions in a conditional setting.

Our paper makes progress towards addressing these challenges on two fronts. First, we develop a methodology to decompose the conditional market risk premium and risk premia on higher-order moments of excess market returns (e.g., the variance risk premium, skewness risk premium, etc.) into sources of risk related to contingent claims on down, up, and moderate market returns. The decomposition requires the assumption of no-arbitrage, but it does not rely on assumptions about the particular functional form of investor preferences nor does it rely on assumptions about the market return distribution. Using this methodology, we estimate conditional contingent claims-based sources of risk premia at each date in our sample. We call these our "data-implied" decompositions. Although an understanding of these decompositions could be useful in many settings, we focus on using our data-implied decompositions as a diagnostic tool to evaluate implications from prominent representative agent models as an application of our methodology. Second, and to this end, we develop a methodology that allows us to estimate analogous decompositions implied by a wide array of prominent representative agent asset pricing models. We call these our "model-implied" decompositions. Comparing the model-implied decompositions to the data-implied decompositions identifies significant discrepancies. Our data-implied decompositions supply a host of new empirical facts that current models fail to explain.

To illustrate our decomposition, we focus on the market risk premium. It is defined as the difference between the expected market return under the physical and risk-neutral measures, $\mathbb{RP} \equiv \mathbb{E}\left[R_M\right] - \mathbb{E}^*\left[R_M\right] = \mathbb{E}\left[\left(R_M - R_f\right)\right]$, where R_f is the risk-free rate. We can use an identity to decompose the total risk premium into components contingent on realized market returns in different regions of the market return space as

$$\mathbb{RP} \;\equiv\; \mathbb{RP}_d + \mathbb{RP}_c + \mathbb{RP}_u \tag{1}$$

where $\mathbb{RP}_s \equiv \mathbb{E}\left[\left(R_M - R_f\right)\mathbb{I}_{A_s}\right] - \mathbb{E}^*\left[\left(R_M - R_f\right)\mathbb{I}_{A_s}\right]$. $\mathbb{E}$ represents the expectation operator under the physical measure, $\mathbb{E}^*$ represents expectation operator under the risk-neutral measure, $\mathbb{I}$ is an indicator function, and A_s represents sets describing different regions of the market return space with $s \in \{d, c, u\}$. d, c, and u represent either down, moderate ("central"), or up market returns, respectively.[1] This decomposition effectively separates the total market risk premium into contributions from risks associated with market return realizations in each of these regions. We call these different components of the risk premium the "downside risk premium", the "central risk premium", and the "upside risk premium", respectively.[2]

[1]This decomposition makes use of the identity $\mathbb{I}_A \equiv \mathbb{I}_{A_d} + \mathbb{I}_{A_c} + \mathbb{I}_{A_u} = 1$ with $A \equiv A_d \cup A_c \cup A_u = \mathbb{R}^+$. That is, A represents the set of non-negative real numbers, which corresponds to the set of permissible gross market return realizations (assuming limited liability). These sets effectively divide the gross market return space into down market return regions, moderate or "central" market return regions, and up market return regions. To be more concrete, assume A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$. Then A_d represents the set of net market returns less than -10%, A_c represents the set of net market returns between -10% and +10%, and A_u represents the set of net market returns above +10%. We can also express these risk premia in integral form as follows: $\mathbb{RP}_s \equiv \int_{A_s}\left[f\left(R_M\right) - f^*\left(R_M\right)\right]\left(R_M - R_f\right) dR_M$ where $f\left(R_M\right)$ and $f^*\left(R_M\right)$ are the market return distributions under the physical and risk-neutral measures, respectively. This formulation makes it clear that our risk premium conforms to the standard definition of the market risk premium as the integrated difference between excess market returns under the physical and risk-neutral measures. When A_s represents the entire market return space, we recover the standard market risk premium. We simply consider integrating over different regions of the market return space, A_s, to compute different components of the risk premium.

[2]We refer to $\mathbb{RP}_s$ in Equation 1 as the "level" of the risk premium associated with region A_s, and $\mathbb{RP}_s/\mathbb{RP}$ as the "contribution" of the risk premium associated with region A_s to the total risk premium.

We estimate the data-implied conditional decomposition and compare it to model-implied conditional decompositions at each date in our sample for a number of prominent representative agent models. Some of our main results are highlighted in Figure 1, which plots the total market risk premium and contributions from each of our three main regions of interest as a fraction of the total risk premium implied by the data, the long-run risk model in Bansal and Yaron (2004) ("BY"), the long-run risk model in Drechsler and Yaron (2011) ("DY"), and the disaster risk model in Wachter (2013) ("Wachter").

The disparate decomposition behavior across panels in Figure 1 makes it clear why having an understanding of conditional risk premia from each of these regions is important when evaluating various models. The total market risk premium levels implied by the data and models in Figure 1 are similar on average, but the conditional behavior varies drastically. The data-implied decomposition (Panel (a)) implies that the downside and central risk premia contribute approximately equal amounts to the total risk premium on average across time, with the upside risk premium contributing a much lower amount. BY (Panel (b)) implies that the central risk premium is the main contributor to the total risk premium, which is inconsistent with the data. DY (Panel (c)) and Wachter (Panel (d)) both imply the downside and central risk premia are major contributors to the total risk premium with the downside contributing less on average in both cases, which is inconsistent with the data.

The DY and Wachter model results highlight the importance of characterizing conditional risk premium behavior when comparing different models. Unconditionally, the average risk premium contributions implied by the DY and Wachter models are similar. However, conditionally, it is clear the time series behavior of the Wachter model is more similar to that implied by the data. In general, we find discrepancies between the market risk premium decompositions implied by the data and all models we investigate. We also perform a decomposition of the conditional variance risk premium and document similar discrepancies.

We now provide some additional details related to our data- and model-implied decompositions. We begin with the data-implied decomposition. Given a no-arbitrage representative

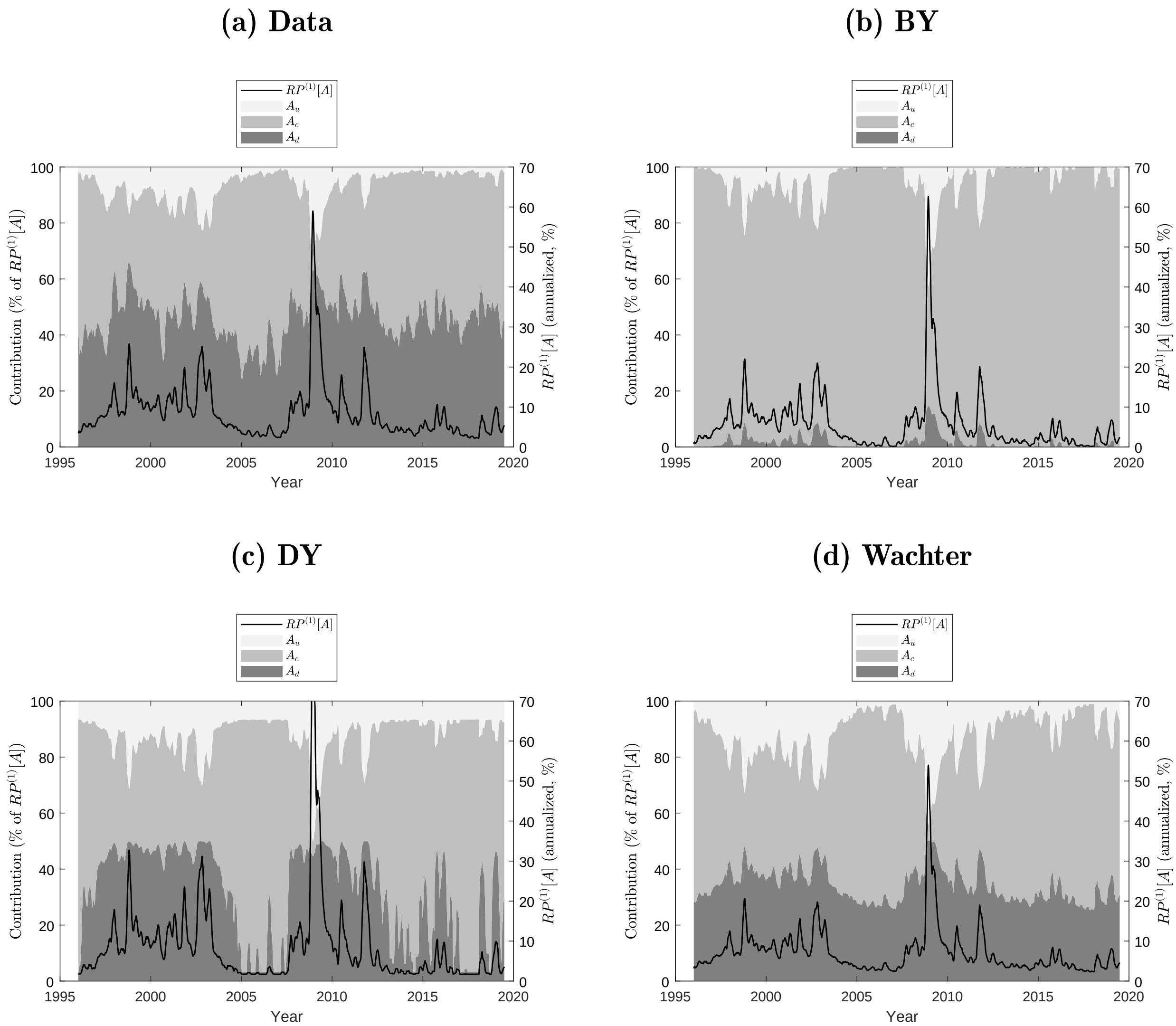

Figure 1
Main Results Comparing Risk Premium Decompositions

These graphs summarize our main market risk premium decomposition results (30-day horizon, annualized) from our data-implied decomposition in Panel (a), the Bansal and Yaron (2004) long-run risks model in Panel (b), the Drechsler and Yaron (2011) long-run risks model in Panel (c), and the Wachter (2013) disaster risk model in Panel (d). Panel (a) uses the data-implied decomposition with estimated preference parameters reported in Table 1, but results are similar when using restricted preference parameters from Subsection 2.3. $\mathbb{RP}^{(1)}[A]$ represents the total risk premium and is measured on the right vertical axes. The dark/medium/light shaded regions represent the downside/central/upside risk premium contributions to the total risk premium (as a fraction of the total risk premium) at each date and are measured on the left vertical axes. These decompositions are computed using A_d=[0, 0.9], A_c=[0.9, 1.1], and A_u=[1.1, +∞). All time series are smoothed by averaging over two months of lagged daily data to reduce the appearance of noise.

agent economy, we derive analytic expressions for each component of the risk premium on arbitrary moments of the excess market return.[3] We show how the required physical moments (i.e., the physical return distribution) can be expressed in terms of risk-neutral moments constructed from option prices and investor preference parameters associated with a generic utility function. We do not require any assumptions about the specific functional form of investor preferences or the market return distribution. Although the decomposition applies to risk premia on arbitrary moments of the excess market return, we put more emphasis on the first moment (i.e., the market risk premium) since it has received the most attention in the literature and is typically the primary object of study in representative agent asset pricing models. We also provide results related to the second moment (i.e., the variance risk premium) since some of the models we investigate were constructed to explain the risk premium associated with this moment. Our data-implied decomposition can be computed at any date given a cross-section of option prices and an estimate of investor preference parameters linked to a generic utility function. We provide two versions of our data-implied decomposition. In the "unrestricted" version, we use data to estimate investor preference parameters then use these estimated preference parameters to construct the decomposition. In the "restricted" version, we provide economic restrictions on preference parameters that allow us to compute the decomposition without the need to estimate preference parameters.

One feature of our data-implied decomposition that is not apparent in Figure 1 is that we can construct the decomposition over different forecasting horizons, which exploits the fact that options have different maturities. For example, the decomposition can be constructed using option prices available at date t with maturity date $t + T_1$, and again using options with maturities at a second date, $t + T_2$ with $T_2 > T_1$. The first decomposition applies to

[3] In the parlance of Equation 1, we derive expressions for risk premium decompositions of the form $\mathbb{RP} \equiv \mathbb{RP}_d + \mathbb{RP}_c + \mathbb{RP}_u$ where $\mathbb{RP}_s \equiv \mathbb{E}\left[(R_M - R_f)\,\mathbb{I}_{A_s}\right] - \mathbb{E}^*\left[(R_M - R_f)\,\mathbb{I}_{A_s}\right]$ (i.e., the market risk premium), and $\mathbb{RP}^{(n)} \equiv \mathbb{RP}_d^{(n)} + \mathbb{RP}_c^{(n)} + \mathbb{RP}_u^{(n)}$ where $\mathbb{RP}_s^{(n)} \equiv \mathbb{E}_t\left[\left(R_{M,t\to T} - \mathbb{E}_t\left[R_{M,t\to T}\right]\right)^n \mathbb{I}_{A_s}\right] - \mathbb{E}_t^*\left[\left(R_{M,t\to T} - \mathbb{E}_t^*\left[R_{M,t\to T}\right]\right)^n \mathbb{I}_{A_s}\right]$ for $n > 1$ (i.e., higher-order risk premia such as the variance risk premium, skewness risk premium, etc.).

expected returns over the period from t to $t+T_1$, whereas the second decomposition applies to expected returns over the period from t to $t+T_2$. We perform decompositions for horizons ranging from 30 to 360 days. In addition to the conditional behavior of the each component of the data-implied market risk premium in Panel (a) of Figure 1, we also find that the level of the total market risk premium (when annualized), $\mathbb{RP}$, and each of its components, $\mathbb{RP}_s$, decrease as the horizon increases. As horizon increases, the contribution of $\mathbb{RP}_c$ to the total risk premium decreases, whereas the contributions of $\mathbb{RP}_d$ and $\mathbb{RP}_u$ increase.

Our framework also allows us to estimate preference parameters (e.g., relative risk aversion, skewness tolerance, and kurtosis tolerance) implied by the data at different horizons and in different regions of the return space (i.e., A_d, A_c, and A_u). Our estimate of relative risk aversion is 1.85 when averaged across all regions and horizons. At each horizon, relative risk aversion is a decreasing function in the return space moving from A_d to A_u. This is consistent with the intuition that investors are more risk averse to bad states of the world and less risk averse to good states of the world. Relative risk aversion is also decreasing in horizon, indicating that investors are more risk averse to short-term risks (e.g., over the next month) than long-term risks (e.g., over the next year year).

Next, we turn to additional details related to our model-implied decomposition. The methodology follows two steps. In the first step, we extract conditional model-implied state variables at each date by requiring each model to match salient asset pricing data (e.g., the log price-dividend ratio and risk-neutral market excess return variance). In the second step, we derive expressions for all model-implied moments needed for our decomposition in terms of the state variables. Given the state variables extracted in the first step, we can compute conditional risk premium decompositions implied by each model at each date and compare them to the data-implied decompositions. As highlighted in Figure 1, we find that model-implied decompositions do not conform well to the data-implied decompositions.

We evaluate prominent examples from three classes of representative agent models that have emerged in the literature to explain the market risk premium: 1. Long-run risk models

(Bansal and Yaron, 2004; Bansal, Kiku, and Yaron, 2012; Drechsler and Yaron, 2011), 2. Habit formation models (Bekaert, Engstrom, and Ermolov, 2020, both with and without preference shocks), and 3. Disaster risk models (Gabaix, 2012; Wachter, 2013). We focus on models that feature time-varying risk premia since we would like to compare conditional implications from these models with those from the data. We do not evaluate representative agent models such as those in Barro (2006), Barro (2009), or Backus, Chernov, and Martin (2011) because these models do not feature time-varying state variables and, hence, imply time-invariant risk premia.

Our paper is related to but distinct from Beason and Schreindorfer (2020). In that paper, the authors use a different methodology to estimate a data-implied *unconditional* market risk premium decomposition as a function of the market return space. They compare this with unconditional market risk premium decompositions implied by a number of representative agent models and, similar to us, find large discrepancies between the decompositions. Our work is complimentary to theirs in that we develop a new methodology to estimate a *conditional* market risk premium decomposition. Our methodology also allows us to generate conditional decompositions for risk premia on higher order moments of excess market returns such as the variance risk premium. One major difference between our results and those in Beason and Schreindorfer (2020) is that their results imply the downside risk premium constitutes approximately 80% of the market risk premium (unconditionally), whereas ours imply that it constitutes only about 46% of the risk premium. This difference could be attributed to the different methodologies employed in each paper, which we discuss further in Internet Appendix IA.8.

Our paper is also related to Bollerslev and Todorov (2011), who estimate conditional downside/upside market and variance risk premia similar to our measures. Their methodology relies on extreme value theory to estimate physical moments using high-frequency market return data. Using extreme value theory limits them to estimating risk premia associated with large positive or negative jumps in the market return space. For instance, their methodology cannot be used to estimate the conditional total risk premium. Contrarily, our methodology does not rely on extreme value theory to estimate physical moments, and can be used to estimate risk premia associated with any region of the market return distribution including the total risk premium. They also rely on approximations that limit their analysis to short horizons, whereas our theory can be used to estimate risk premia at any horizon (given a sufficient cross-section of options).

The remainder of our paper is organized as follows. Section 1 provides the theoretical foundations for our data-implied risk premium decomposition and Section 2 estimates the decomposition empirically. Section 3 develops the methodology that allows us to estimate the decomposition for various representative agent models and presents related empirical results. Section 4 concludes.

1 A Decomposition of Conditional Risk Premia

We derive our main theoretical conditional risk premium decomposition results in this section by expanding on the methodology developed in Chabi-Yo and Loudis (2020). Our theory involves a fair amount of notation, which we define as it is introduced. However, for ease of interpretation a summary can also be found in Internet Appendix IA.1. We begin by deriving an expression for the stochastic discount factor (SDF) in terms of a Taylor expansion of a representative agent's generic utility function. We then show how this can be used to construct the risk premia on arbitrary moments of excess market returns in our decomposition.

1.1 An Expression for the SDF Under Generic Utility

Consider a representative agent with initial wealth W_t and a well-behaved utility function $U(\cdot)$ over terminal wealth $W_T = W_t \omega_t' \mathbf{R}_{t\to T}$, where $\mathbf{R}_{t\to T}$ is a vector of risky asset gross returns, $R_{i,t\to T}$, with $i = 1, ..., n$ and ω_t is a vector of portfolio weights. We assume the utility function $U(\cdot)$ is concave and admits finite higher-order derivatives. The representative agent maximizes expected utility over terminal wealth, $\mathbb{E}_t\left[U\left(W_T\right)\right]$, with first-order conditions given by

$$\mathbb{E}_t\left[U^{'}\left(W_T\right)\left(R_{i,t\to T}-R_{f,t\to T}\right)\right]=0. \tag{2}$$

We assume a risk-free asset exists with gross return denoted by $R_{f,t\to T}$. Note that these first-order conditions apply to any asset i, including the market return, $R_{M,t\to T}$. Assuming the market value is a proxy for the agent's wealth[4] and no-arbitrage conditions hold, the first-order condition in Equation 2 implies the inverse SDF has the form

$$\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}}=\frac{\frac{1}{U^{'}(W_t x_T)}}{\mathbb{E}_t^{*}\left[\frac{1}{U^{'}(W_t x_T)}\right]}, \tag{3}$$

where we define $x_T\equiv R_{M,t\to T}$ for simplicity of notation. Our main goal is to decompose risk premia into contributions associated with the downside region of the market return space (left tail), the center region, and upside region (right tail). To achieve this, we start by partitioning the gross market return space into three subsets defined by A_d, A_c, and A_u such that $A_d\cap A_c\cap A_u=\emptyset$ and $A_d\cup A_c\cup A_u=\mathbb{R}^{+}$. Note that $\mathbb{R}^{+}$ is the set of non-negative real numbers and represents the feasible set for the gross market return space (assuming limited liability). Colloquially, A_d represents the set of down market returns; A_c represents the set of moderate or "central" market returns; and A_c represents the set of up market returns. To be more concrete about the definitions of A_d, A_c, and A_u, consider two constants $\underline{x}$ and $\overline{x}$ satisfying the restriction $\underline{x}<1<\overline{x}$. We can then define the following sets:

$$A_d \equiv \left\{x:0\leq x<\underline{x}\right\}, \tag{4}$$

$$A_c \equiv \left\{x:\underline{x}\leq x<\overline{x}\right\}, \tag{5}$$

$$A_u \equiv \left\{x:x\geq\overline{x}\right\},\ \text{and} \tag{6}$$

$$A \equiv A_d\cup A_c\cup A_u. \tag{7}$$

[4]This is a common assumption in the related literature (see, e.g., Chabi-Yo and Loudis, 2020 and Martin, 2017)

We define A above as the entire gross return space for notation purposes. Next, we decompose the inverse SDF in Equation 3 into three components:

$$\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}} = \frac{\frac{1}{U'(W_t x_T)}}{\mathbb{E}_t^*\left[\frac{1}{U'(W_t x_T)}\right]}\mathbb{I}_{A_d} + \frac{\frac{1}{U'(W_t x_T)}}{\mathbb{E}_t^*\left[\frac{1}{U'(W_t x_T)}\right]}\mathbb{I}_{A_c} + \frac{\frac{1}{U'(W_t x_T)}}{\mathbb{E}_t^*\left[\frac{1}{U'(W_t x_T)}\right]}\mathbb{I}_{A_u}, \tag{8}$$

where $\mathbb{I}$ is an indicator function such that $\mathbb{I}_{A_s} = 1$ if $x_T \in A_s$ and $\mathbb{I}_{A_s} = 0$ otherwise, for $s \in \{d, c, u\}$. Consider three different points in the return space, x_d, x_c, and x_u, such that $x_d \in A_d$, $x_c \in A_c$, and $x_u \in A_u$. We can multiply each component in the right-hand side of Equation 8 by $U'\left(W_t x_s\right)/U'\left(W_t x_s\right)$ (using each component's respective x_s values) to obtain the equivalent decomposition

$$\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}} = \sum_{s\in\{d,c,u\}} \frac{\frac{U'(W_t x_s)}{U'(W_t x_T)}}{\mathbb{E}_t^*\left[\frac{U'(W_t x_s)}{U'(W_t x_T)}\right]}\mathbb{I}_{A_s}. \tag{9}$$

Next, denote

$$g_{x_s}\left(x\right) = \frac{f_{x_s}\left(x\right)}{\mathbb{E}_t^*\left[f_{x_s}\left(x\right)\right]}, \text{ where } f_{x_s}\left(x\right) = \frac{U'\left(W_t x_s\right)}{U'\left(W_t x\right)}, \tag{10}$$

We can use a Taylor series expansion to express $f_{x_s}\left(x\right)$ as

$$f_{x_s}\left(x\right) = 1 + \sum_{k=1}^{\infty}\theta_k\left(x_s\right)\left(x - x_s\right)^k, \tag{11}$$

where

$$\theta_k\left(x_s\right) = \frac{1}{k!}\left(\frac{\partial^k f_{x_s}\left(x\right)}{\partial^k x}\right)\Bigg|_{x=x_s}. \tag{12}$$

Equation 11 allows us to express the inverse SDF in Equation 9 as a piecewise function of $f_{x_s}\left(x\right)$ in the three different regions of interest. Furthermore, we show that $\theta_k\left(x_s\right)$ (for $k \in \{1, 2, 3\}$) can be expressed as

$$\theta_1\left(x_s\right) = \frac{1}{x_s\tau\left(x_s\right)},\ \theta_2\left(x_s\right) = \frac{\left(1 - \rho\left(x_s\right)\right)}{x_s^2\tau^2\left(x_s\right)}, \text{ and } \theta_3\left(x_s\right) = \frac{\left(1 - 2\rho\left(x_s\right) + \kappa\left(x_s\right)\right)}{x_s^3\tau^3\left(x_s\right)} \tag{13}$$

in the Appendix A ("Proof for Equation 13"), where we use the following standard preference parameter definitions:

$$\tau(x_s) \equiv -\frac{U'(W_t x_s)}{W_t x_s U''(W_t x_s)}, \tag{14}$$

$$\rho(x_s) \equiv \frac{1}{2!}\frac{U'''(W_t x_s)\, U'(W_t x_s)}{\left(U''(W_t x_s)\right)^2}, \text{ and} \tag{15}$$

$$\kappa(x_s) \equiv \frac{1}{3!}\frac{U''''(W_t x_s)\left(U'(W_t x_s)\right)^2}{\left(U''(W_t x_s)\right)^3}. \tag{16}$$

As with the $\theta_k(x_s)$, the preference parameters are functions of the return space with $x_s \in \{x_d, x_c, x_u\}$. $\tau(x_s)$ is a measure of risk tolerance ($1/\tau(x_s)$ is a measure of relative risk aversion), $\rho(x_s)$ is a measure of skewness tolerance, and $\kappa(x_s)$ is a measure of kurtosis tolerance. Note that all parameters in Equations 14-16 are positive if the agent's utility function conforms to standard preference theory.[5] To the extent these preference parameters are functions of W_t, they are time-varying. For tractability, we assume they are constant over time in our empirical estimation. Note that fixing $\tau(x_s)$ to be constant does not imply our representative investor has CRRA utility.[6]

Next, we provide an alternative expansion of the inverse of the SDF centered around $R_{f,t\to T}$ rather than three different values of x_s.[7] The binomial theorem implies the following

[5]That is, the preference parameters are positive if Sign $\left[U^{(k)}(\cdot)\right] = (-1)^{k+1}$ where $U^{(k)}(\cdot)$ represents the k-th derivative of the utility function (see Eeckhoudt and Schlesinger, 2006; Deck and Schlesinger, 2014; and Noussair, Trautmann, and VanDeKuilen, 2014). See Chabi-Yo, Leisen, and Renault (2014) for additional details related to the interpretation of these preference parameters.

[6]CRRA utility implies $-U'(W_t R_{M,t\to T})/[W_t R_{M,t\to T} U''(W_t R_{M,t\to T})]$ is constant for all potential realizations of future wealth, $W_T = W_t R_{M,t\to T}$. CRRA utility also imposes restrictions on higher order preference parameters such as ρ and κ given a particular choice for relative risk aversion to achieve constant relative risk aversion regardless of future wealth. For instance, if the representative investor were to have CRRA utility with risk aversion $\gamma = 1/\tau$, this implies the skewness tolerance must be given by $\rho = (1+\gamma)/(2\gamma)$, which we do not impose. Furthermore, we assume $-U'(W_t x_s)/[W_t x_s U''(W_t x_s)]$ is constant in each region given current wealth. By not imposing CRRA parameter restrictions on $\rho(x_s)$ and $\kappa(x_s)$ given $\tau(x_s)$, we do not have a relative risk aversion that is constant for all potential realizations of future wealth, $W_T = W_t R_{M,t\to T}$ as is the case for CRRA utility. We present results in which we assume the representative investor has CRRA utility (along with other common utility specifications) in Internet Appendix IA.3 and show that the empirical decompositions are drastically different than those in our main results.

[7]This will allow us to construct risk premium expressions in terms of risk-neutral moments of excess market returns only (as opposed to moments of returns less each of the three x_s values) while also allowing

exact identity:

$$\left(x-x_s\right)^k \equiv \sum_{j=0}^{k} \frac{k!}{j!\,(k-j)!} \left(R_{f,t\to T}-x_s\right)^j \left(x-R_{f,t\to T}\right)^{k-j}. \tag{17}$$

We replace $\left(x-x_s\right)^k$ in Equation 11 with the equivalent expression in Equation 17 to obtain

$$f_{x_s}\left(x\right) = 1 + \sum_{k=1}^{\infty} \theta_k\left(x_s\right) \sum_{j=0}^{k} \frac{k!}{j!\,(k-j)!} \left(R_{f,t\to T}-x_s\right)^j \left(x-R_{f,t\to T}\right)^{k-j}.$$

We can then express the inverse SDF as

$$\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}} = \sum_{s\in\{d,c,u\}} g_{x_s}\left(x_T\right) \mathbb{I}_{A_s}, \tag{18}$$

where

$$g_{x_s}\left(x_T\right) = \frac{1 + \sum_{k=1}^{\infty} \theta_k\left(x_s\right) \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \left(-1\right)^j \left(x_s-R_{f,t\to T}\right)^j \left(x-R_{f,t\to T}\right)^{k-j}}{1 + \sum_{k=1}^{\infty} \theta_k\left(x_s\right) \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \left(-1\right)^j \left(x_s-R_{f,t\to T}\right)^j \mathbb{E}_t^*\left(x_T-R_{f,t\to T}\right)^{k-j}}. \tag{19}$$

The inverse SDF in Equation 18 is a composite of Taylor expansions from the three regions of interest in the market return space.

Working with the inverse SDF expression in Equation 18 can be motivated in two ways. First, we would like to decompose conditional risk premia into components associated with different regions of interest in the return space. As we will see below, the decompositions in each region depend on the SDF in that region. Second, Taylor series expansions around a given point are only accurate for observations that lie in the neighborhood of that point. This is a potentially important consideration when modeling the SDF as a function of market returns, where having different Taylor expansions for different regions of the return space can improve the accuracy of the approximated SDF. Equivalently, this is an important consideration with regards to investor preferences if investors have different attitudes towards the risks associated with these different regions. This setup allows us to estimate different pref-

us to use region-specific preference parameters as defined in Equations 14-16.

erence parameters associated with different regions of the market return space and estimate a region-specific SDF.

1.2 Conditional Physical Moments of the Excess Market Return

We define conditional truncated risk-neutral and physical moments of the excess market return as

$$\mathbb{M}^{*(n)}_{t\to T}\left[A_s\right] \equiv \mathbb{E}^*_t\left[\left(R_{M,t\to T} - R_{f,t\to T}\right)^n \mathbb{I}_{A_s}\right] \text{ and} \tag{20}$$

$$\mathbb{M}^{(n)}_{t\to T}\left[A_s\right] \equiv \mathbb{E}_t\left[\left(R_{M,t\to T} - R_{f,t\to T}\right)^n \mathbb{I}_{A_s}\right] \tag{21}$$

for $A_s \in \{A, A_d, A_c, A_u\}$. When $A_s = A$, these are simply regular (untruncated) moments (i.e., $\mathbb{I}_A \equiv 1$ since A represents the set of all feasible gross market returns). When $A_s \in \{A_d, A_c, A_u\}$, these are truncated moments. In the case of untruncated moments, we occasionally denote $\mathbb{M}^{(n)}_{t\to T}\left[A\right] = \mathbb{M}^{(n)}_{t\to T}$ and $\mathbb{M}^{*(n)}_{t\to T}\left[A\right] = \mathbb{M}^{*(n)}_{t\to T}$ for brevity.

We now show how the inverse SDF expression in Equation 18 can be used to construct physical moments of excess market returns from risk-neutral moments. For any $n > 0$, the conditional physical moments of excess returns on any asset i can be expressed using the identity

$$\begin{aligned}
\mathbb{M}^{(n)}_{i,t\to T}\left[A_s\right] &\equiv \mathbb{E}_t\left[\frac{M_{t\to T}}{\mathbb{E}_t\left[M_{t\to T}\right]}\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}}\left(R_{i,t\to T} - R_{f,t\to T}\right)^n \mathbb{I}_{A_s}\right] \\
&= \mathbb{E}^*_t\left[\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}}\left(R_{i,t\to T} - R_{f,t\to T}\right)^n \mathbb{I}_{A_s}\right].
\end{aligned} \tag{22}$$

$\mathbb{M}^{(n)}_{i,t\to T}\left[A_s\right]$ describes untruncated moments when $A_s = A$ and truncated moments when $A_s \in \{A_d, A_c, A_u\}$. We can view $M_{t\to T}/E_t\left[M_{t\to T}\right]$ as the Radon-Nikodym derivative for a change of measure between the physical and risk-neutral distributions. Assuming no-arbitrage, we use the Radon-Nikodym theorem to move from the first to second equality in Equation 22. This equation can be written equivalently as

$$\mathbb{M}^{(n)}_{i,t\to T}\left[A_s\right] - \mathbb{M}^{*(n)}_{i,t\to T}\left[A_s\right] = \mathbb{COV}^*_t\left[\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}}, \left(R_{i,t\to T} - R_{f,t\to T}\right)^n \mathbb{I}_{A_s}\right]. \tag{23}$$

We can think of $\mathbb{M}^{(n)}_{i,t\to T}[A_s] - \mathbb{M}^{*(n)}_{i,t\to T}[A_s]$ as the moment risk premium on an asset that pays returns equal to $(R_{i,t\to T} - R_{f,t\to T})^n \mathbb{I}_{A_s}$.[8] Equation 23 is very intuitive. It says that the risk premium is equal to the *risk-neutral covariance* between the *inverse SDF* and an asset's return. This is a risk-neutral counterpart to the fundamental asset pricing equation's implication that the risk premium on any asset's return is the *negative physical covariance* between the *SDF* and the asset's return.

Next, we specialize i in Equation Equation 22 to be the market return, and replace $E_t[M_{t\to T}]/M_{t\to T}$ with the expression in Equation 18. This allows us to derive an analytic expression for physical moments on the excess market returns in terms of risk-neutral moments in the following proposition.

Proposition 1. *Assuming no arbitrage, conditional physical moments on excess market returns obey the exact decomposition*

$$\mathbb{M}^{(n)}_{t\to T}[A_s] = \mathbb{M}^{*(n)}_{t\to T}[A_s] + \frac{\sum_{k=1}^{\infty} \theta_k(x_s)\, \mathbb{COV}^*_t\left[(R_{M,t\to T} - x_s)^k, (R_{M,t\to T} - R_{f,t\to T})^n \mathbb{I}_{A_s}\right]}{1 + \sum_{k=1}^{\infty} \theta_k(x_s)\, \mathbb{E}^*_t\left[(R_{M,t\to T} - x_s)^k\right]} \tag{24}$$

for $A_s \in \{A, A_d, A_c, A_u\}$ defined in Equations 4-7.

Proof. See Appendix A. ∎

While Proposition 1 allows us to express conditional truncated moments in terms of truncated risk neutral covariances, Corollary 1 shows how the conditional physical truncated moments can be expressed in terms of conditional truncated risk neutral moments.

Corollary 1. *Assuming no arbitrage, conditional physical moments on excess market returns*

[8] When $n = 1$ this risk premium just becomes the standard expression for the risk premium on the return of any asset, i, when $A_s = A$ (or the risk premium on truncated moments of the excess return when $A_s \neq A$). If we further specialize i to be the market, then this becomes the market risk premium (or truncated versions of the market risk premium when $A_s \neq A$).

obey the exact decomposition:

$$\mathbb{M}_{t\to T}^{(n)}\left[A_s\right]=\mathbb{M}_{t\to T}^{*(n)}\left[A_s\right]+\frac{\sum_{k=1}^{\infty}\sum_{j=0}^{k}\lambda_t\left(x_s,k,j\right)\left(\mathbb{M}_{t\to T}^{*(n+k-j)}\left[A_s\right]-\mathbb{M}_{t\to T}^{*(k-j)}\left[A\right]\mathbb{M}_{t\to T}^{*(n)}\left[A_s\right]\right)}{1+\sum_{k=1}^{\infty}\sum_{j=0}^{k}\lambda_t\left(x_s,k,j\right)\mathbb{M}_{t\to T}^{*(k-j)}\left[A\right]},\quad(25)$$

for $A_s \in \{A, A_d, A_c, A_u\}$ *where*

$$\lambda_t\left(x_s,k,j\right)\equiv\frac{k!\theta_k\left(x_s\right)\left(-1\right)^j\left(x_s-R_{f,t\to T}\right)^j}{j!\left(k-j\right)!}.\quad(26)$$

The parameters $\theta_k\left(x_s\right)$ *are defined in Equation 12 and the sets* A, A_d, A_u, *and* A_c *are defined in Equations 4-7.*

Proof. See Appendix A. ■

Corollary 1 relates physical truncated moments of excess market returns to their risk-neutral counterparts and investor preferences without making any assumptions about the precise form of investor utility. If the risk-neutral quantities and preference parameters are known, Corollary 1 can be used to compute physical truncated moments of excess market returns without relying on any assumptions about the market return distribution, investor utility, or economic fundamentals. Truncated conditional moments can be computed for each region of interest by setting $s = d$, c, or u. We can also derive the following corollary as a special case when $x_s = R_{f,t\to T}$.

Corollary 2. *Assuming no arbitrage and setting* $x_s = R_{f,t\to T}$, *conditional physical moments on excess market returns obey the exact decomposition:*

$$\mathbb{M}_{t\to T}^{(n)}\left[A_s\right]=\mathbb{M}_{t\to T}^{*(n)}\left[A_s\right]+\frac{\sum_{k=1}^{\infty}\theta_k\left(R_{f,t\to T}\right)\left(\mathbb{M}_{t\to T}^{*(n+k)}\left[A_s\right]-\mathbb{M}_{t\to T}^{*(k)}\left[A\right]\mathbb{M}_{t\to T}^{*(n)}\left[A_s\right]\right)}{1+\sum_{k=1}^{\infty}\sum_{j=0}^{k}\theta_k\left(R_{f,t\to T}\right)\mathbb{M}_{t\to T}^{*(k)}\left[A\right]},\quad(27)$$

for $A_s \in \{A, A_d, A_c, A_u\}$. *The parameters* $\theta_k\left(R_{f,t\to T}\right)$ *are defined in Equation 12 and the sets* A, A_d, A_u, *and* A_c *are defined in Equations 4-7.*

Proof. This follows directly from Corollary 1 by setting $x_s = R_{f,t\to T}$. ∎

Corollary 2 will be useful when we restrict preference parameters to be the same across all regions in our restricted risk premium decomposition. Next, we show how untruncated conditional expected excess market return moments can be expressed in terms of truncated conditional expected excess return moments.

Proposition 2. *Assuming no arbitrage, conditional total excess market return moments obey the exact relationship*

$$\mathbb{M}^{(n)}_{t\to T}\left[A\right] = \sum_{s\in\{d,c,u\}} \mathbb{M}^{(n)}_{t\to T}\left[A_s\right], \tag{28}$$

where $\mathbb{M}^{(n)}_{t\to T}\left[A_s\right]$ *is defined in Equation 21 and the sets* A*,* A_d*,* A_u*, and* A_c *are defined in Equations 4-7.*

The proof of Proposition 2 follows by taking the expected value of the exact decomposition $\left(R_{M,t\to T} - R_{f,t\to T}\right)^n \equiv \sum_{s\in\{d,c,u\}} \left(R_{M,t\to T} - R_{f,t\to T}\right)^n \mathbb{I}_{A_s}$.

1.3 Conditional Risk Premium Decomposition

We now have the necessary machinery to construct our conditional risk premium decomposition for moments of excess market returns.

Definition 1. *We define the conditional risk premium on the n-th order truncated excess market return moment as*

$$\mathbb{RP}^{(n)}_{t\to T}\left[A_s\right] \equiv \begin{cases} \mathbb{E}_t\left[\left(R_{M,t\to T} - R_{f,t\to T}\right)\mathbb{I}_{A_s}\right] - \mathbb{E}^*_t\left[\left(R_{M,t\to T} - R_{f,t\to T}\right)\mathbb{I}_{A_s}\right] & \text{for } n=1 \\ \mathbb{E}_t\left[\left(R_{M,t\to T} - \mathbb{E}_t\left[R_{M,t\to T}\right]\right)^n\mathbb{I}_{A_s}\right] - \mathbb{E}^*_t\left[\left(R_{M,t\to T} - \mathbb{E}^*_t\left[R_{M,t\to T}\right]\right)^n\mathbb{I}_{A_s}\right] & \text{for } n>1. \end{cases} \tag{29}$$

The expressions in Definition 1 hold for $A_s \in \{A, A_d, A_c, A_u\}$. When $A_s = A$, these represent untruncated risk premia; when $A_s \in \{A_d, A_c, A_u\}$, these represent truncated risk premia. When $n = 1$ and $A_s = A$, the standard market risk premium expression obtains. When $n > 1$, this relationship describes standard definitions for the variance risk premium ($n = 2$), the skewness risk premium ($n = 3$), the kurtosis risk premium ($n = 4$), and so on.[9,10] These truncated risk premia represent compensation for exposure to risk associated with contingent claims in each of these regions of the market return space. We refer to the risk premia associated with A, A_d, A_c, and A_u in Equation 30 as the total risk premium, the downside risk premium, the central risk premium, and the upside risk premium, respectively.[11] Finally, we express these risk premia in terms of physical and risk-neutral moments in the following proposition.

Proposition 3. *The total risk premium* $\mathbb{RP}^{(n)}_{t\to T}[A]$ *can be decomposed into terms related to truncated risk premia as*

$$\mathbb{RP}^{(n)}_{t\to T}[A] = \sum_{s\in\{d,c,u\}} \mathbb{RP}^{(n)}_{t\to T}[A_s] \tag{30}$$

for any n. *When* $n = 1$, $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ *can be computed directly using physical moments from Corollary 1 as*

$$\mathbb{RP}^{(1)}_{t\to T}[A_s] \quad = \quad \mathbb{M}^{(1)}_{t\to T}[A_s] - \mathbb{M}^{*(1)}_{t\to T}[A_s] \tag{31}$$

for $A_s \in \{A, A_d, A_c, A_u\}$. *When* $n > 1$, *the risk premium on moments of the excess market return can be written as*

$$\mathbb{RP}^{(n)}_{t\to T}[A_s] = \sum_{k=0}^{n} \frac{n!\,(-1)^{n-k}}{(n-k)!k!} \mathbb{M}^{(k)}_{t\to T}[A_s] \left(\mathbb{M}^{(1)}_{t\to T}[A]\right)^{n-k} - \mathbb{M}^{*(n)}_{t\to T}[A_s] \tag{32}$$

[9] For example, when $n = 2$ we can show that $\mathbb{RP}^{(n)}_{t\to T}[A] = \mathbb{VAR}(R_{M,t\to T}) - \mathbb{VAR}^*(R_{M,t\to T})$ (i.e., the physical minus risk-neutral market return variance), consistent with standard definitions of the variance risk premium.

[10] With a slight abuse of nomenclature, we refer to the third and fourth excess market returns as "skewness" and "kurtosis", respectively.

[11] The downside, central, and upside risk premia are always with respect to a particular choice of n in Equation 30. For instance, the downside risk premium component of the market risk premium ($n = 1$) is not the same as the downside risk premium component of the variance risk premium ($n = 2$). We do not explicitly designate this in the nomenclature of the different components. However, it should be clear in context when these components are related to the market or variance risk premium.

for $A_s \in \{A, A_d, A_c, A_u\}$. The sets A, A_d, A_u, and A_c are defined in Equations 4-7.

Proof. See Appendix A. ■

Note that risk premia and moments expressed as functions of A represent untruncated risk premia and moments, whereas those expressed as functions of A_s represent truncated risk premia and moments. This proposition shows how the risk premia on arbitrary moments of the excess market return can be can be expressed in terms of physical and risk-neutral truncated moments. Corollaries 1 and 2 allow us to express these risk premia entirely in terms of risk-neutral moments on excess market returns, which can be estimated using option price data.

2 Data-Implied Risk Premium Decompositions

In this section, we estimate the market (variance) risk premium decomposition described in Proposition 3, which obtains when $n = 1$ ($n = 2$). We estimate this decomposition at five horizons: 30, 60, 90, 180, and 360 calendar days.

The decomposition depends on having an estimate of the preference parameters $\tau(x_s)$, $\rho(x_s)$, and $\kappa(x_s)$. We approach this task in two ways in the main text. First, we use data to estimate preference parameters described in Equations 14, 15, and 16 in sample to generate an "unrestricted" market risk premium decomposition. To do this, we exploit the link between physical and risk-neutral market excess return moments provided by Corollary 1. In particular, we use market return data to proxy for the physical moments and S&P 500 options data to construct proxies for the risk-neutral moments to estimate the preference parameters. Given estimates of the the preference parameters, we can use risk-neutral moments to compute implied physical moments according to Corollary 1. We then construct conditional truncated risk premia according to Proposition 3. Second, we simply choose preference parameters ex ante based on preference parameter restrictions used in the restricted lower bound of Chabi-Yo and Loudis (2020) to generate a "restricted" version of the market risk premium decomposition. As a third alternative, we also derive closed-form expressions for the decomposition and estimate it assuming the representative agent has standard preferences (e.g., log, CRRA, CARA, and HARA). We relegate this analysis to Internet Appendix IA.3 for brevity.[12]

2.1 Data

In order to use Corollary 1 to estimate preference parameters $\tau(x_s)$, $\rho(x_s)$, and $\kappa(x_s)$, we need estimates of risk-neutral moments of the excess market return. We use S&P 500 index option data to compute all risk-neutral moments (truncated and untruncated). Given a set of option prices on a cross-section of strikes, we use the the Carr and Madan (2001) spanning formula to construct the risk-neutral moments described in Equation 20. In theory, we need to integrate functions of options prices over a continuous set of strikes to compute these measures. In practice, we accomplish this by curve fitting implied volatility curves at each maturity and performing numerical integration using the corresponding implied option prices

[12]We find that the implied risk premium levels under these utility specifications can be quite different than those from our main data-implied decompositions. However, contributions to the total risk premium are actually more similar to those from the data-implied decomposition compared to those from the representative agent model-implied decompositions we study in Section 3. The additional structure imposed by the representative agent models can lead to model misspecification that has counterfactual implications for the relative contributions to the total risk premium to which the utility-based decompositions are immune. Note that assuming a specific functional form for utility places restrictions on the SDF, but does not place restrictions on the market return distribution or state variables that describe the economy as is the case with the representative agent models.

using the trapezoidal rule.[13,14] To mitigate estimation noise but balance this with timeliness of information in the risk-neutral moments, we use a lagged five-trading day moving average of the raw moments when estimating preference parameters and to construct our risk premium decomposition. More details on risk-neutral moment construction can be found in the Internet Appendix IA.4.

Data is from Option Metrics, is daily, and spans January, 1996 to June, 2019. We apply standard filters on the options data before constructing risk-neutral moments.[15] When constructing risk-neutral excess return moments, we use risk-free rates implied by the Option Metrics Zero Curve data and obtain the S&P 500 Index price from Option Metrics. We construct risk-neutral moments at fixed horizons (30, 60, 90, 180, and 360 days) by computing risk-neutral moments using options at observed horizons and extrapolating (or interpolating) to the desired horizon. We cannot construct reliable measures of the necessary risk-neutral moments at longer horizons due to limitations on options availability.

S&P 500 return data are from CRSP. We use ex-dividend returns since the risk-neutral

[13] Fitting implied volatility curves for the purposes of computing risk-neutral moments implied by options prices has become standard in the literature (see Jackwerth and Cuesdeanu, 2018). For other examples of this approach in practice, see Chang, Christoffersen, and Jacobs (2013), Carr and Wu (2009), Jiang and Tian (2005), and Ait-Sahalia and Lo (1998). We use piecewise cubic Hermite interpolating polynomials over observed strikes and assume that implied volatilities above (below) the highest (lowest) observed strikes are constant and take on the values corresponding to the highest (lowest) observed strikes. We perform the numerical integration over a strike price-to-index price range of 0.01 to 2 with 1,000 grid points over this range. We also construct risk-neutral moments by integrating over observed prices directly in Internet Appendix IA.2.3 and find that this does not alter our results qualitatively, nor does it alter them much quantitatively.

[14] One might be concerned that our measures and conclusions are distorted due to potential options mispricing that induces excessive skewness in the implied volatility smirk (for instance, due to demand-driven price pressure as described in Garleanu, Pedersen, and Poteshman, 2009). We explore a method to correct the effect of such mispricing in Internet Appendix IA.2.4 and find that this does not materially alter our main results or conclusions.

[15] In particular, we: 1. Delete all options that are not monthlies (i.e., weeklies, quarterlies, and EOM options); 2. Delete all duplicate records based on same date, expiration date, strike, and option type (we keep the duplicate with either the most recent trade date, or, if the duplicate has the same most recent trade date, the record with the highest volume); 3. Delete all options with a closing bid of $0; 4. Delete options with maturities less than seven days; 5. Delete call options with recorded prices higher than the index price and put options with prices less than the strike price times the risk-free bond price; 6. Delete all options with bid prices higher than ask prices; and 7. Remove options that violate convexity restrictions.

moments are constructed from European options. Therefore, the risk-neutral moment expressions derived in the Internet Appendix describe risk-neutral moments on ex-dividend returns. Returns are daily and range from January, 1926 through December, 2019.[16] Excess returns at each horizon are computed by compounding daily returns to the horizon of interest and subtracting the compounded risk-free rate obtained from Kenneth French's website.

2.2 Unrestricted Risk Premium Decomposition

We need to estimate the preference parameters $\tau(x_s)$, $\rho(x_s)$, and $\kappa(x_s)$ at three points in the market return space corresponding to $s \in \{d, c, u\}$. These preference parameters are required to compute the physical moments (via Corollary 1) needed to implement the risk premium decomposition in Proposition 3.

Given realized excess market returns as proxies for physical moments and risk-neutral moments estimated from options data, we estimate the preference parameters $\tau(x_s)$, $\rho(x_s)$, and $\kappa(x_s)$ using non-linear weighted least squares to minimize the squared error implied by Corollary 1 when $n = 1$, 2, and 3. We use daily data and estimate parameters separately for the horizons, T, of interest: 30, 60, 90, 180, and 360 calendar days.[17] We set $\underline{x} = 0.9$ and $\overline{x} = 1.1$ with $x_d = 0.85$, $x_c = 1$, and $x_d = 1.15$ in all reported results. That is, we are interested in studying risk premia associated with down market returns less than -10%, central market returns between -10% and +10%, and up market returns greater than +10%. Note that we must estimate a total of nine preference parameters for each horizon of interest: three parameters ($\tau(x_s)$, $\rho(x_s)$, and $\kappa(x_s)$) for each of the three regions of interest ($s \in \{d, c, u\}$) in the return space. See Internet Appendix IA.5 for more details related to this estimation.

Table 1 provides preference parameter estimates in each region and at each horizon of

[16] We use returns before 1996 in forecasting regressions to construct out-of-sample R-squared values according to Goyal and Welch (2008).

[17] In our estimation, these preference parameters will be a function of horizon, T, but we suppress this dependence in their notation for simplicity.

interest.[18] Values in brackets represent 95% confidence intervals based on block-bootstrapped estimates and indicate all parameter estimates are statistically significant at the 95% level. We also provide estimates of the relative risk aversion, which is given by $1/\tau(x_s)$ according to Equation 14. Relative risk aversion is generally decreasing with horizon and in region order (from the downside region to the upside region). Averaging across all horizons, the average relative risk aversion is 2.87 in the down region, 1.74 in the central region, and 0.94 in the up region. These results imply investors are more averse to downside risk than to upside risk. Averaging across regions, relative risk aversion is nearly monotonically decreasing in horizon. It starts at a value of 2.21 at the 30-day horizon and rises to a value of 2.39 at the 60-day horizon before monotonically decreasing to 0.86 at the 1-year horizon. These results imply investors are more averse to short-term risks than to long-term risks.

Similar patterns hold when considering risk tolerance, $\tau(x_s)$, skewness tolerance, $\rho(x_s)$, and kurtosis tolerance, $\kappa(x_s)$. Since risk tolerance is the inverse of relative risk aversion, its patterns are reversed relative to those discussed above but the interpretation is the same. $\rho(x_s)$ is increasing in region s. Investors are more tolerant to positive skewness than negative skewness. It converges to values of around two with increasing horizon, which implies investors are more tolerant to skewness at longer horizons. $\kappa(x_s)$ is also increasing in region, implying investors are more tolerant to fat tails in the the upper part of the return distribution and less tolerant to those in the lower part. $\kappa(x_s)$ converges to approximately four with increasing horizon, implying that, like with $\rho(x_s)$, investors have approximately the same tolerance for higher moment risk at longer horizons whether it comes from the down or up region.[19,20]

[18] We provide corresponding preference parameter plots in the Internet Appendix IA.2.1 (Figures IA.1 and IA.2) for visualization.

[19] Note that our estimated preference parameters conform to sign restrictions on investor utility from standard economic theory (e.g., Sign $\left[U^{(k)}(\cdot)\right] = (-1)^{k+1}$; see Eeckhoudt and Schlesinger, 2006; Deck and Schlesinger, 2014; and Noussair, Trautmann, and VanDeKuilen, 2014).

[20] Our estimated preference parameters also pin down the conditional SDF in different regions of the market return space according to Equation 18. We provide results related to the implied conditional SDF in Internet Appendix IA.2.1.

Figure 2 provides plots of the data-implied market risk premia (i.e., when $n = 1$ in Equation 29) when preference parameters from Table 1 are used to construct physical moments according to Corollary 1.[21] Panels (a), (b), and (c) plot the risk premium decomposition in levels, and Panels (d), (e), and (f) plot their contributions to the total risk premium as a fraction of the total risk premium for horizons of 30, 90, and 360 days, respectively. It is interesting to note that upside and downside risk premia typically increase conditionally at the same times, but this is not always the case.

Focusing on the 30-day horizon, the downside risk premium contributes at least 20% of the total market risk premium regardless of the calendar date. This highlights the fact that investor concerns about crashes or disasters do not vanish during periods of low market volatility. The downside risk premium contribution varies over time and increases drastically during crisis periods. For instance, it increases to over 60% during major crisis periods such as the collapse of Long Term Capital Management in 1998, the March 2000 Dot-com bubble, the September 11 terrorist attacks in 2001, the 2002 stock market downturn, the bankruptcy of Lehman Brothers in 2008, and the 2010 flash crash.

Table 2 provides summary statistics for the data-implied market risk premium decomposition constructed using preference parameters from Table 1. All risk premia are annualized for comparability across different horizons. Panels A and B provide summary statistics for the decomposition in levels and as fractions of the total market risk premium, respectively. We begin by focusing on the "Unconditional Statistics" in Panel A, which are averaged over our full sample. The average annualized 30-day (360-day) risk premium during our sample period is 8.72% (4.44%), which is higher (lower) than the average annualized 30-day (360-day) S&P

[21] In principle, we could construct total risk premia implied by the unrestricted decomposition using a set of preference parameters from any of the three regions of interest. In practice, we compute all three sets of total risk premia and report results for total risk premia using the average of these three time series. The time series are very similar, and using any one (as opposed to the average) does not change our main results and conclusions. We apply this procedure when reporting both the total market and variance risk premium results.

500 ex-dividend excess returns over the same period, which is 5.99% (5.92%).[22] The average risk premium is monotonically decreasing in horizon (excluding the 30-day horizon) and its standard deviation is decreasing in horizon.

On average, the downside risk premium is the largest risk premium across all horizons. It is approximately decreasing in horizon, ranging from 5.71% at the 60-day horizon to 3.03% at the 360-day horizon. The average upside risk premium is the lowest of all risk premia at the 30-day horizon (0.97%). It is concave in horizon, rising to a value of 1.64% at the 180-day horizon before falling to 1.16% at the 360-day horizon. The central risk premium decreases with horizon from a value of 3.33% at the 30-day horizon to a value of 0.19% at the 360-day horizon. These results imply investors demand a large premium for exposure to downside risk compared to that associated with central or upside risk.

We also provide average risk premia conditional on risk-neutral variance of the excess return at the 30-day horizon, $\mathbb{M}^{*(2)}_{t\to T}[A]$, in the columns labeled "Conditional Means" in Table 2. The "Lo" column corresponds to average risk premia conditional on $\mathbb{M}^{*(2)}_{t\to T}[A]$ being below its first quartile; the "Mid" column corresponds to $\mathbb{M}^{*(2)}_{t\to T}[A]$ falling between its first and third quartiles; and the "Hi" column corresponds to $\mathbb{M}^{*(2)}_{t\to T}[A]$ falling above its third quartile. These ranges correspond to periods of low, moderate, and high market volatility. The total average risk premium level is increasing in $\mathbb{M}^{*(2)}_{t\to T}[A]$ across all time horizons. That is, the total risk premium is higher during periods of high market volatility. This pattern also holds for the upside risk premia, but these do not increase as much as the downside risk premia when volatility increases. The average central risk premium is increasing in market volatility at short horizons and decreasing at longer horizons.

Next, we turn to Panel B in Table 2, which provides summary statistics on the time series of each risk premium as a fraction of the total risk premium. We refer to these as

[22]Since we include a constant term in the preference parameter estimation (see Equation IA.44) but not when computing the physical moments used in our risk premium measures (Corollary 1), our average risk premium need not match the ex post observed average excess market return.

risk premium contributions. The results are largely in line with those presented for the risk premium levels in Panel A.[23] The downside risk premium constitutes the largest fraction of the total risk premium unconditionally (across time). Its contribution is increasing in both horizon and with market volatility. The upside risk premium constitutes the smallest fraction of the total risk premium. Its contribution also increases with horizon and market volatility. The central risk premium constitutes a large fraction of the total risk premium at short horizons, but this decreases and contributes almost nothing to the total risk premium at the 360-day horizon. The central risk premium constitutes a large fraction of the total risk premium during low-volatility periods, but this contribution decreases substantially with increasing horizon and volatility.

Table 3 provides results from forecasting regressions of the form

$$R_{M,t\to T} - R_{f,t\to T} \quad = \quad a_T + b_T \mathbb{RP}^{(1)}_{t\to T}\left[A_s\right] + \varepsilon_{t\to T} \tag{33}$$

for horizons of 30, 90, and 360 days. The left portion of the table provides results including all data and the right portion of the table removes data from the 2008 Financial Crisis (August, 2008 through January, 2009) since this period represents a significant outlier for realized market returns in our relatively short sample period. If we have reasonable measures for the total market risk premium, $\mathbb{RP}^{(1)}_{t\to T}\left[A\right]$, it should forecast realized excess market returns. In particular, we expect $a_T = 0$ and $b_T = 1$ when $A_s = A$. We also explore the ability of the truncated risk premia to forecast market returns, although in these cases there are no clear implications for the values of either a_T or b_T.

We cannot reject the individual null hypotheses that $a_T = 0$ or $b_T = 1$ in the full sample using the total risk premium. However, it is slightly disconcerting that the b_T coefficients are not statistically significant at short horizons. This is likely a symptom of sample selection

[23] Note that the mean contributions in Panel B of Table 2 do not necessarily equal the fraction of mean level risk premia to the total risk premium in Panel A due to Jensen's inequality effects. That is, Panel B reports summary statistics for the time-varying fractions, not fractions based on the unconditional summary statistics reported in Panel A.

related to including the financial crisis period, which ex post was a period over which realized returns were low for an extended period of time. The fact that the 360-day forecasting regression yields a statistically significant b_T estimate gives some support for this interpretation. Additionally, when we remove six months of crisis period data from the regression, coefficients become statistically significant at all horizons. We cannot reject either of the individual null hypothesis in these cases. We also provide pseudo-out-of-sample R-squared statistics computed according to the methodology in Goyal and Welch (2008). In particular, these statistics are computed using our raw total market risk premium as a forecasting variable compared to the alternate model that uses historical average S&P 500 return as a forecasting variable. We call these "pseudo-out-of-sample" R-squared values since they use preference parameters estimated from our full sample to construct our risk premium measures. The main message is similar to that from the regression results. The R-squared values indicate the risk premium does a relatively poor job forecasting excess returns at short horizons in the Full Sample relative to the historical average excess return (i.e., the R-squared values are negative at the 30- and 90-day horizons). However, the R-squared values are positive in the Full Sample at the 360-day horizon (0.02) and even larger in the Ex Crisis sample (increasing up to 0.09). Results using the truncated risk premia are similar.

Figure 3 provides plots of the data-implied variance risk premium decomposition (i.e., when $n = 2$ in Equation 29) when preference parameters from Table 1 are used to construct physical moments according to Corollary 1. Panels (a), (b), and (c) plot the risk premium decomposition in levels, and Panels (d), (e), and (f) plot them in terms of the fraction of the overall risk premium for horizons of 30, 90, and 360 days, respectively. The downside risk premium is the main contributor to the total variance risk premium over all horizons and the upside risk premium becomes a larger contributor as the horizon increases.

Table 2 provides summary statistics for the data-implied variance risk premium decomposition using preference parameters from Table 1. Panels A and B provide summary statistics for the decomposition in levels and as fractions of the total variance risk premium, respectively. We begin by focusing on the "Unconditional Statistics" in Panel A, which are averaged over our full sample. The average annualized 30-day (360-day) variance risk premium during our sample period is -1.03% (-1.54%). These are similar in magnitude to variance risk premium estimates reported in Dew-Becker et al. (2017) and Bekaert, Engstrom, and Ermolov (2020), which were approximately -1.5%[24] and -1.9%[25] at the monthly horizon, respectively. Conditionally, the variance risk premium increases during periods of high volatility, which implies investors are willing to pay relatively more for insurance against volatility during turbulent times. This finding is consistent with those in Dew-Becker et al. (2017), who use different data (variance swap contracts) to characterize the variance risk premium.

2.3 Restricted Risk Premium Decomposition

One might be concerned that our measures in the previous sub-section are overly-reliant on in-sample estimates of the preference parameters. To mitigate this concern, in this sub-section we impose the same restrictions on our preference parameters as those used by Chabi-Yo and Loudis (2020) to construct a restricted lower bound on the market risk premium. Namely, we set $\tau = 1$, $\rho = 2$, and $\kappa = 4$ across all regions and horizons.[26] We use Corollary 2 to compute the physical moments using risk-neutral moments since we only have one set of preference parameters for the whole return space in this case.[27] The restricted preference

[24]This is based on Figure 2 in Dew-Becker et al. (2017). Specifically, one can approximate the implied variance risk premium (annualized and in percent) by taking $100 \times 12 \left(\mathbb{E}\left[F_t^0\right] - \mathbb{E}\left[F_t^1\right]\right)$ using their nomenclature with $100 \times \sqrt{12 \times \mathbb{E}\left[F_t^0\right]} \approx 17.4$ and $100 \times \sqrt{12 \times \mathbb{E}\left[F_t^1\right]} \approx 21.2$ according to the figure.

[25]This is based on Table 8 in Bekaert, Engstrom, and Ermolov (2020). The annualized value reported herein takes their reported monthly value (0.0016) and multiplies it by -1, 12, and 100. The -1 is to account for the fact that they define the variance risk premium to be the risk-neutral minus physical variance, the 12 is to annualize it, and the 100 is to give it the interpretation of being in percent. Note that they define the variance risk premium using log market returns whereas we use simple market returns, which could lead to some discrepancies between the measures.

[26]See Chabi-Yo and Loudis (2020) for more details on these restrictions. We also note here that Noussair, Trautmann, and VanDeKuilen (2014) provide evidence that $\tau < 1$, $\rho > 1$, and $\kappa > 1$ in a different empirical setting, which imply that our choices for the restricted τ, ρ, and κ yield a lower bound on the market risk premium.

[27]Technically, with respect to the nomenclature in Corollary 2, we set $\tau\left(R_{f,t\to T}\right) = 1$, $\rho\left(R_{f,t\to T}\right) = 2$, and $\kappa\left(R_{f,t\to T}\right) = 4$.

parameter values are similar to the values of $\tau(x_s)$, $\rho(x_s)$, and $\kappa(x_s)$ reported in Table 1 averaged over all horizons and regions, which are 0.73, 2.88, and 2.84, respectively. Under standard preference assumptions, the market risk premium is decreasing in τ, increasing in ρ, and decreasing in κ, so we expect the average market risk premium constructed using these restricted parameters to be lower than that constructed using the estimated parameters and reported in Table 2.

Table 5 reports summary statistics when the risk premium decomposition is constructed using the restricted preference parameters. Results are generally in line with those reported using estimated preference parameters in Table 2 except for one notable exception: average total risk premia are lower (higher) at short (long) horizons than the related values in Table 2.[28] We provide the associated risk premium decomposition plots in Internet Appendix IA.2.2 (Figure IA.4) for brevity since they are visually similar to the unrestricted decomposition plots in Figure 2. Forecasting regression results using the risk premia from the restricted decomposition are similar to those from our main unrestricted decomposition and are reported in Internet Appendix IA.2.2 for brevity.

3 Model-Implied Risk Premium Decompositions

In this section, we consider one application of our data-implied decomposition: using it as a diagnostic tool to assess whether various prominent representative agent asset pricing models generate decompositions that are consistent with our data-implied decomposition. To this end, we develop a methodology for computing the risk premium decomposition implied by representative agent models in the asset pricing literature. We focus on three long-run risk models (Bansal and Yaron, 2004, Bansal, Kiku, and Yaron, 2012, and Drechsler and Yaron, 2011), two habit formation models (Bekaert, Engstrom, and Ermolov, 2020 with and

[28]This is because: 1. The restricted τ is typically higher (lower) than the estimated value at short (long) horizons, 2. The restricted ρ is typically lower (higher) than the estimated value at short (long) horizons, and 3. The restricted κ is typically higher (lower) than the estimated value at short (long) horizons.

without preference shocks), and two disaster risk models (Gabaix, 2012 and Wachter, 2013). We show that a projection of the SDF implied by a representative agent model can be recast as a function of aggregate wealth in Internet Appendix IA.6. Although our model-implied decompositions do not rely on this result, it does justify using our data-implied decomposition (which projects the SDF onto aggregate wealth) to evaluate representative agent models. In all cases, we evaluate models using the original parameterizations reported in each respective reference. We report market risk premium decomposition results for all models and variance risk premium decomposition results for all models except the first two long-run-risk models (Bansal and Yaron, 2004 and Bansal, Kiku, and Yaron, 2012), which feature normal shocks and were not intended to target the variance risk premium.

Our model-implied decomposition methodology is straightforward and involves two steps. First, we extract model-implied state variables by matching model-implied asset pricing moments to those observed in the data at each date. Under each model we can express asset pricing moments as linear functions of state variables.[29] Conversely, given N observed asset pricing moments at any date, we can exactly identify implied state variables in a model having N state variables. The maximum number of state variables among all models we consider is three. We therefore identify a consistent set of three salient asset pricing moments from the data to use for extracting state variables implied by each model: (1) the log-price-dividend ratio, (2) risk-neutral excess market return variance, and (3) risk-neutral excess market return skewness. We proxy for the last two moments using our $\mathbb{M}^{*(2)}_{t\to T}$ and $\mathbb{M}^{*(3)}_{t\to T}$ measures, respectively, with 30-day horizons since all models were originally calibrated at the monthly frequency. We proxy for the log-price-dividend ratio using Shiller's CAPE index,

[29] One exception is for the Gabaix (2012) model. We describe the state variable extraction process for this model in more detail in its respective section.

which we refer to as $\log(P_t/E_t)$.[30,31] Given these asset pricing moments from the data, we transform each to have the same sample mean as the unconditional model-implied values when extracting state variables implied by each model. This transformation ensures that the extracted state variables have (approximately) the same average values in our sample as their unconditional values implied by the original model calibrations. It also ensures that models imply approximately the same average risk premia in our sample as implied by the original model calibrations.

For the interested reader, summary statistics for the extracted state variables from all models can be found in Internet Appendix Table IA.5. We also provide summary statistics for the model-implied values based on original model calibrations with 95% confidence intervals that we would expect to observe given our sample length under each model's null. Average extracted state variable values are similar in magnitude to and fall within the confidence intervals implied by the calibrated models. This is expected given our state variable extraction methodology. Although the state variable extraction methodology imposes that we match model-implied state variable first moments, it does not impose restrictions related to other moments. For instance, our extracted state variables are typically more volatile than those implied by the calibrated state variable dynamics from each model, which implies our model-implied risk premium decompositions will be more volatile than those implied by the original model calibrations.

[30] Note that all models we consider are consumption-based models without production. In such models, dividend cash flows are more analogous to earnings in the data rather than dividends, which is why we choose $\log(P_t/E_t)$ in the data as a proxy for $\log(P_t/D_t)$ in the models. This approach was also used in Wachter (2013) to extract the implied conditional state variable in that model.

[31] For consistency, when evaluating models with one, two, or three state variables we always use $\left[\mathbb{M}^{*(2)}_{t\to T}\right]$, $\left[\mathbb{M}^{*(2)}_{t\to T}, \log(P_t/E_t)\right]$, or $\left[\mathbb{M}^{*(2)}_{t\to T}, \log(P_t/E_t), \mathbb{M}^{*(3)}_{t\to T}\right]$, respectively, as the asset pricing moments to match in the state variable extraction procedure. Clearly, we could have chosen to use any observable asset pricing moments but chose these due to their theoretical connections to risk premia and salience in the asset pricing literature. The only models with a one state variable are those from Gabaix, 2012 and Wachter, 2013. We use $\mathbb{M}^{*(2)}_{t\to T}$ as the single asset pricing moment for state variable extraction since these models have difficulty matching the high price-dividend ratios observed in the late 1990s and early 2000s. Results from using $\log(P_t/E_t)$ are available upon request.

Second, noting that risk premia in each model are just functions of the model's state variables, we can compute the model-implied risk premium decomposition given the extracted state variables. All derivations and technical results related to the state variable extraction procedure and risk premium decomposition calculations for each model are provided in Internet Appendix IA.7. We provide only the final decomposition results here for brevity.

3.1 Long Run Risk Models

In this subsection, we consider the class of models in which the representative agent has recursive preferences as in Epstein and Zin (1989). Specifically, we estimate our decomposition for models in Bansal and Yaron (2004), Bansal, Kiku, and Yaron (2012), and Drechsler and Yaron (2011).[32]

3.1.1 Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012)

Bansal and Yaron (2004) propose an economic mechanism that relies on long-run risk to explain key stylized empirical asset pricing facts. Bansal, Kiku, and Yaron (2012) extend the Bansal and Yaron (2004) model by allowing consumption shocks to affect the dividend process.[33] Both models include two state variables (x_t and σ_t^2), so we use $\log(P_t/E_t)$ and $\mathbb{M}_{t\to T}^{*(2)}$ to extract implied state variables at each date for both models (independently) using Result IA.1. We use Results IA.2 and IA.3 to compute the implied physical and risk-neutral moments, respectively. See Internet Appendix IA.7.3.1 for additional details.

Panels (a) and (d) ((b) and (e)) in Figure 4 plot the risk premium decompositions for the Bansal and Yaron (2004) (Bansal, Kiku, and Yaron, 2012) model. Compared to the data-implied 30-day horizon decompositions (Figure 2, Panels (a) and (d)), both the Bansal and

[32]We also report results for Bollerslev, Tauchen, and Zhou (2009) in Internet Appendix IA.7.3.2, but leave them out of the main text due to the fact that this model is known to generate implausible risk premia (see, for instance, Bekaert, Engstrom, and Ermolov (2020) for further documentation of this fact).

[33]The model in Bansal, Kiku, and Yaron (2012) was designed to highlight important differences in the asset pricing implications of the long run risk model relative to the habit formation model in Campbell and Cochrane (1999).

Yaron (2004) and Bansal, Kiku, and Yaron (2012) decompositions imply that the central risk premium comprises a larger fraction of the overall risk premium than in the data-implied decomposition. Consequently, the upside and downside risk premia implied by these models comprise a smaller fraction of the overall risk premium than in the data-implied decomposition. The discrepancies are a consequence of the fact that these models are conditionally log-normal and therefore do not often generate realized returns in regions of the return space for which we define downside and upside risk. Interestingly, the unconditional contribution of downside and upside risk is not (effectively) zero, as implied by the unconditional models and documented by Beason and Schreindorfer (2020). This is related to the fact that our state variables are more volatile than those implied by the calibrated state variable processes in each of these models, which occasionally widens the return distribution enough so that the downside and upside risk premia contribute to the overall risk premium in a non-negligible manner.[34]

Table 6 provides summary statistics for these market risk premium decompositions. The average total risk premia implied by the data for Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012) are 5.66% and 6.75%, respectively, which are similar to model-implied values based on simulation (5.55% and 6.67%, respectively).[35] The conditional means of the risk premium decomposition measures imply that all components of the risk premium are increasing with market volatility, which is what we observe in the data. However, as implied by the risk premium plots, the central risk premium comprises the majority of the total risk premium in both models, which is inconsistent with observations from the data-implied

[34]It is also interesting to note that the upside risk premium is larger in magnitude than the downside risk premium. If we were concerned with the risk premium on log-returns, we would expect the downside and upside risk premia to be symmetric. However, since we use simple returns this induces positive skewness in the simple return distribution relative to the log return distribution, causing the upside risk premium to be larger in magnitude than the downside risk premium in these models. This is counterfactual to what we observe in the data.

[35]This is true for all models we investigate since we transform the log P_t/E_t and risk-neutral moments from the data to have the same unconditional means as the log P_t/D_t and risk-neutral moments implied by each model for the purposes of state variable extraction.

decomposition.

Table 8 provides the average differences between the data-implied market risk premia and those implied by Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012). The data-implied total, downside, and upside risk premia are significantly larger than those implied by the models. The data-implied central risk premium is significantly lower than that implied by the models.

3.1.2 Drechsler and Yaron (2011)

Drechsler and Yaron (2011) extend the Bansal and Yaron (2004) model by allowing for jumps in both consumption growth and its volatility. Doing so allows them to better match the mean, volatility, skewness, and kurtosis of consumption growth and stock market returns observed in the data. Their model also generates a variance risk premium that forecasts market excess returns, which is a key stylized fact in the data. The model includes three state variables (x_t, $\overline{\sigma}_t^2$ and σ_t^2), so we use $\log\left(P_t/E_t\right)$, $\mathbb{M}^{*(2)}_{t\to T}$, and $\mathbb{M}^{*(3)}_{t\to T}$ to extract implied state variables at each date using Result IA.7. We use Results IA.8 and IA.9 to compute the implied physical and risk-neutral moments, respectively. See Internet Appendix IA.7.3.3 for additional details.

Panels (c), and (f) in Figure 4 plot the market risk premium decompositions. The downside risk premium is typically a large contributor to the total risk premium. However, during periods of low volatility the central risk premium becomes the dominant contributor to the total risk premium. Panels (a) and (d) in Figure 7 plot the variance risk premium decomposition, which shows that the downside risk premium effectively constitutes the entire variance risk premium except during periods when the total risk premium is small.

Table 6 (Table 7) provides summary statistics for the market (variance) risk premium decomposition. Conditionally, the average the downside market and variance risk premia increase in magnitude as risk-neutral volatility increases, similar to the data-implied results. Table 8 provides the average difference between the data-implied risk premia and those implied by the Drechsler and Yaron (2011) model, and shows that the contributions of the downside/central/upside market (variance) risk premia are significantly lower/higher/higher (lower/lower/higher) than those implied by the data.

3.2 Habit Formation Models: Bekaert, Engstrom, and Ermolov (2020)

Bekaert, Engstrom, and Ermolov (2020) and Bekaert and Engstrom (2017) develop a new class of habit formation models that aim to better capture features of macroeconomic variables.[36] The Bekaert and Engstrom (2017) model requires a computationally intensive numerical solution procedure, but Bekaert, Engstrom, and Ermolov (2020) propose a more tractable version of the model that is able to explain key stylized asset pricing facts while retaining the desirable features of the consumption growth distribution featured in Bekaert and Engstrom (2017). We therefore focus on the habit formation model from Bekaert, Engstrom, and Ermolov (2020) in our analysis.

The Bekaert, Engstrom, and Ermolov (2020) model has two variants: one with preference shocks and one without preference shocks. The model without preference shocks has two state variables (q_t and n_t), so we use $\log(P_t/E_t)$ and $\mathbb{M}^{*(2)}_{t\to T}$ to extract implied state variables for this model at each date. The model with preference shocks has three state variables (q_t, n_t, and s_t), so we use $\log(P_t/E_t)$, $\mathbb{M}^{*(2)}_{t\to T}$, and $\mathbb{M}^{*(3)}_{t\to T}$ to extract implied state variables at each date. We can use Result IA.10 for the state variable extraction in both cases. We use Results IA.11 and IA.12 to compute the implied physical and risk-neutral moments, respectively. See Internet Appendix IA.7.4 for additional details.

Panels (a) and (d) ((b) and (e)) in Figure 5 plot the market risk premium decompositions under the model with (without) preference shocks. The models with and without preference shocks yield decompositions with quite different implications regarding which regions of the return space contribute most to the total risk premium. In the case of the model without pref-

[36] Their models represent an improvement on the habit formation model in Campbell and Cochrane (1999). In particular, their setup allows them to better match observed consumption growth skewness in the data.

erence shocks (Panel (d)) it is clear that the downside risk premium is the largest contributor to the the total risk premium, consistently contributing approximately 80%. The central risk premium typically contributes between approximately 15%-20% to the total risk premium. In the case of the model with preference shocks (Panel (c)), the central risk premium is typically the largest contributor ranging between approximately 60-80%. The downside risk premium contribution ranges between approximately 20-60% and occasionally contributes more than the central risk premium. In both cases, the upside risk premium provides only a minor contribution to the total risk premium.

Comparing these results to those implied by the data at the 30-day horizon (Figure 2, Panels (a) and (d)), the model without preference shocks matches many key features of the data. The contribution of the central risk premium is relatively large during low market volatility periods, but is outweighed by the contribution of the downside risk premium during high market volatility periods. The downside and upside risk premia also typically increase in tandem in both the data- and the model-implied decompositions.

Table 6 provides summary statistics for implied market risk premium decomposition. The average total risk premia implied by the data for the models with and without preference shocks are 4.84% and 5.92%, respectively, which are similar to model-implied values based on simulation (4.26% and 5.56%, respectively). The conditional means of the market risk premium decomposition measures imply that all components of the risk premium are increasing with market volatility, which is what we observe in the data. The downside risk premium comprises the majority of the total risk premium in the model without preference shocks, which is inconsistent with observations from the data-implied decomposition. However, the contributions and time series behavior of the downside, upside, and central risk premia in the model with preference shocks are qualitatively more similar to the data-implied decomposition than for other models we study.

Table 8 provides the average difference between the data-implied risk premia and those implied by the models with and without preference shocks. The data-implied total, downside, central, and upside risk premia are significantly larger than those implied by the models in level (Panel A). This holds in all cases except for the downside risk premium in the model without preference shocks, which is significantly lower than what the data implies. In terms of the contribution to total risk premium (Panel B), the data implies a central risk premium contribution that is significantly lower than that implied by model with preference shocks. The data also implies downside risk premium contribution that is significantly lower than that implied by the model with preference shocks. In all other cases, the data implies contributions that are higher than those implied by the models.

Figure 7 provides plots for the variance risk premium decomposition for the model with preference shocks. It shows that the primary contributor is the downside risk premium, although there are periods where the central risk premium becomes a sizable contributor. Plots for the variance risk premium decomposition from the model without preference shocks are similar so we omit them for brevity. In this case, the central (downside) risk premium contributes less (more) to total risk premium than in the model with preference shocks. Table 7 provides summary statistics for the implied variance risk premium decompositions and Table 8 provides the average difference between the data-implied risk premia and those implied by the Bekaert, Engstrom, and Ermolov (2020) models. Table 8 shows that the contributions of the downside/central/upside variance risk premia are significantly lower/higher/higher (higher/lower/higher) in the model with (without) preference shocks than those implied by the data.

3.3 Disaster Risk Models

3.3.1 Gabaix (2012)

Gabaix (2012) develops a time-varying disaster risk model that is able to quantitatively explain many standard asset pricing puzzles. We follow the Dew-Becker et al. (2017) implementation (including their parameter choices) in our analysis.[37] In this model, the log price-dividend ratio is non-linear in the model's single state variable, L_t. Therefore, we solve for $\mathbb{M}^{*(2)}_{t\to T}$ as a function of the state variable numerically given the assumed L_t process conditionally as a function of L_t. Given an estimate for the the model-implied $\mathbb{M}^{*(2)}_{t\to T}$ as a function of L_t, we extract the implied L_t at each date by matching the transformed $\mathbb{M}^{*(2)}_{t\to T}$ from the data as usual. Given the extracted state variable, we estimate physical and risk-neutral moments needed for our decomposition at each date via simulation according to expressions for these moments in Internet Appendix IA.7.5.1.

Panels (a) and (c) in Figure 6 plot the market risk premium decomposition. The downside risk premium is consistently the largest contributor to the total risk premium with the upside risk premium contributing essentially nothing. Table 6 (Table 7) provides summary statistics for the implied market (variance) risk premium decomposition. Conditionally, both the average downside market and variance risk premia increase in magnitude as risk-neutral volatility increases, which is similar to the data-implied results. Table 8 provides the average difference between the data-implied risk premia and those implied by the Gabaix (2012) model. The contributions of the downside/central/upside market (variance) risk premia are significantly higher/lower/lower (higher/lower/higher) than those implied by the data.

3.3.2 Wachter (2013)

Wachter (2013) develops a time-varying disaster risk model that aims to explain the excess volatility puzzle. We follow the Dew-Becker et al. (2017) discretization of Wachter (2013) (including their parameter choices) since this provides a convenient version of the model that is calibrated at the monthly frequency. The model includes one state variable, λ_t, so we use $\mathbb{M}^{*(2)}_{t\to T}$ to extract the implied state variable at each date using Result IA.15. We then use

[37] This calibration is similar to that in Gabaix (2012) and is also able to match the Sharpe ratio of one-month variance swaps reported in Dew-Becker et al. (2017). We use this version of the model since we are also interested in evaluating the model's ability to match the conditional variance risk premium and its decomposition.

Results IA.14 and IA.15 to compute the implied physical and risk-neutral moments needed for the risk premium decompositions. See Internet Appendix IA.7.3.2 for additional details.

Panels (b) and (d) in Figure 6 plot the market risk premium decomposition. The central risk premium contributes the majority of the total risk premium. The downside risk premium contributes less than the central risk premium, but more than the upside risk premium. Interestingly, the time variation in these decomposition contributions are similar to those from the data (see Figure 2, Panel (d)). Panels (c) and (f) in Figure 7 plot the variance risk premium decomposition. It is clear from this figure that the downside risk premium effectively constitutes the entire variance risk premium.

Table 6 (Table 7) provides summary statistics for the implied market (variance) risk premium decomposition and Table 8 provides the average difference between the data-implied risk premia and those implied by the Wachter (2013) model. The contributions of the downside/central/upside market (variance) risk premia are significantly lower/higher/higher (higher/lower/lower) than those implied by the data.

4 Conclusions

In this paper, we propose a novel methodology that allows us to decompose the risk premium on arbitrary moments of excess market returns into components related to compensation for exposure to the left tail, the center, and the right tail of the market return distribution at each date. Under the assumption of no-arbitrage, we derive analytic expressions for each of these components in terms of option prices without making any assumptions about the market return distribution or the functional form of investor preferences. This allows us to quantify the contributions from conditional downside, central, and upside risk premia to the conditional total risk premium at any date and across investment horizons ranging from one month to one year. We provide empirical results for to two special cases where we estimate decompositions for the market risk premium and the variance risk premium.

The downside risk premium comprises a large fraction of the total market risk premium across all dates and horizons. The central risk premium comprises a large fraction of the total market risk premium as well, but its contribution decreases with both horizon and in risk (as measured by market volatility). The upside risk premium contributes less to the total market risk premium unconditionally than the first two, but, like the downside risk premium, its contribution increases with horizon and when market volatility increases.

The downside risk premium also comprises a large fraction of the total variance risk premium regardless of horizon. The central risk premium comprises a small fraction of the total variance risk premium, and its contribution decreases with horizon. The upside risk premium also comprises a small fraction of the total variance risk premium, and its contribution increases with horizon.

These data-implied decompositions provide powerful tools for understanding risk premium dynamics. Although these decompositions may be useful in many settings, we choose to use them to evaluate prominent representative agent asset pricing models to highlight one application. A common feature among all these models is that they (typically) have state variables that vary over time, leading to time-varying risk premia. Despite this, the success of a model is often judged based on its ability to match unconditional moments in the data. Part of the hurdle to evaluating these models conditionally has been identifying a consistent method to link state variables to observable information. Another hurdle has been the difficulty of measuring conditional risk premia in the data that can be used to evaluate implications from the models. We overcome the latter hurdle with our data-implied decomposition. We overcome the former by developing a consistent (across models) methodology for extracting conditional state variables given salient asset pricing data.

We identify clear inconsistencies between the model- and data-implied decompositions. First, imposing that a model match the conditional log price-dividend ratio and risk-neutral moments of the excess market return typically yields extracted state variable time series having higher volatility than that implied by assumptions in the original models. This implies that more effort should be placed in ensuring that modeling assumptions with respect to state variable processes are consistent with their data-implied counterparts. Second, we find statistically significant differences between the data-implied and model-implied decompositions across almost all models and components of the market and variance risk premia. This is particularly true of log-normal models, which have a difficult time matching the total downside risk premium implied by the data, but it is also true of models with shocks that produce higher tail risk via non-Gaussian shocks.

Admittedly, moments from these decompositions were not targeted in the original calibrations of any of the models we investigate and represent high hurdles for parsimonious representative agent models to tackle. However, the reasonably good performance of the Bekaert, Engstrom, and Ermolov (2020) and Wachter (2013) models provides hope that such models will be up to the challenge of explaining conditional risk premia and their components. We hope our data-implied decomposition provides a useful tool for calibrating similar models in the future, and helps provide a deeper understanding the sources of conditional risk premia.

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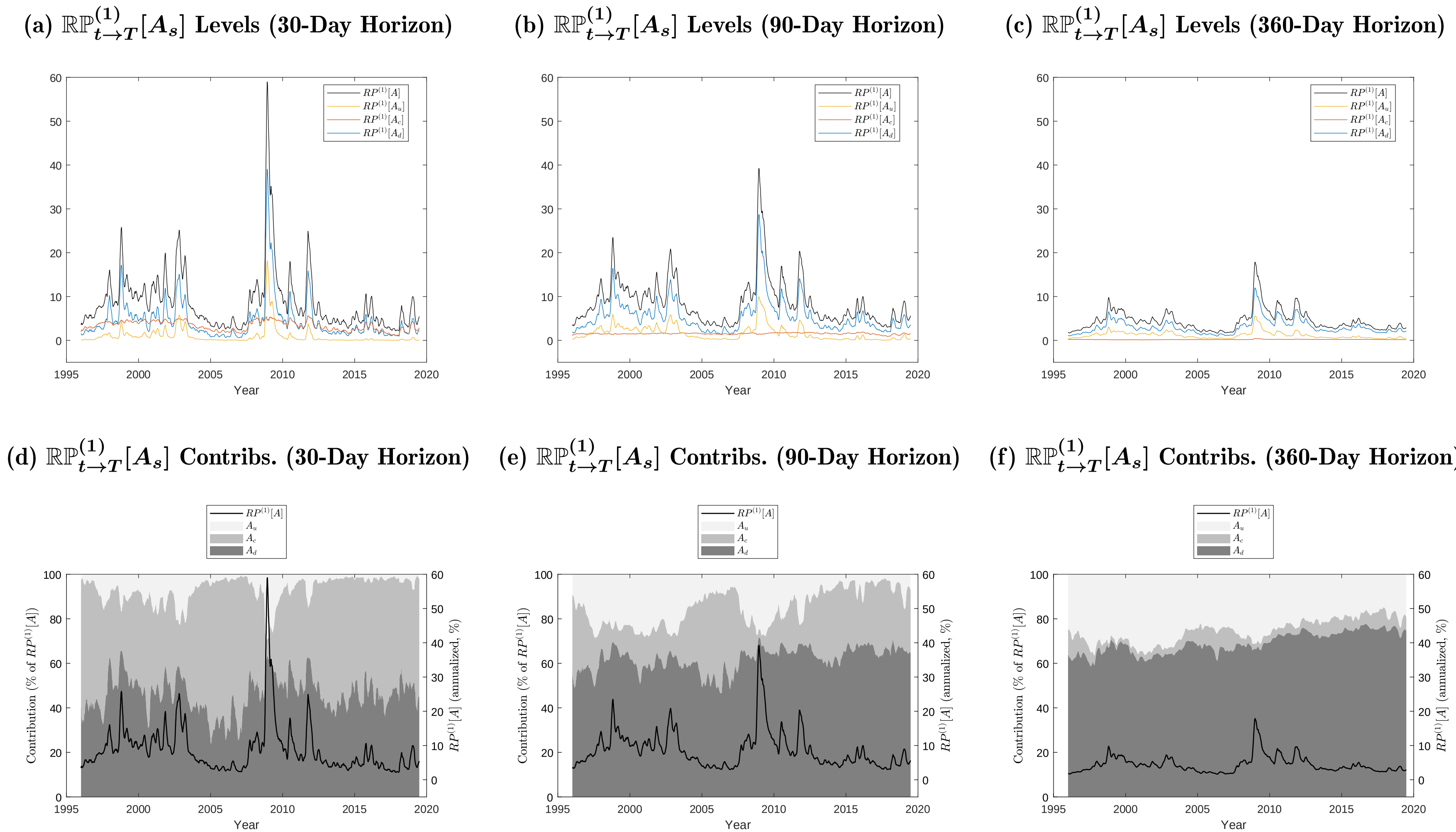


(a) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Levels (30-Day Horizon)
(b) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Levels (90-Day Horizon)
(c) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Levels (360-Day Horizon)
(d) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Contribs. (30-Day Horizon)
(e) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Contribs. (90-Day Horizon)
(f) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Contribs. (360-Day Horizon)

Figure 2

Data-Implied Market Risk Premium Decomposition (Unrestricted Preference Parameters)

These graphs plot the data-implied unrestricted risk premium decompositions from Section 2.2 based on Proposition 3 with $n = 1$ (i.e., the market risk premium). The decompositions use preference parameters reported in Table 1. Panels (a)-(c) plot the annualized risk premium levels at each date in percent. Panels (d)-(e) plot each component's contribution to the total risk premium at each date as a fraction of the total risk premium. The dark/medium/light shaded regions represent the downside/central/upside risk premium contributions, respectively, and are measured on the left vertical axes. The decompositions are computed at three horizons (30 days - Panels (a) and (d); 90 days - Panels (b) and (e); and 360 days - Panels (c) and (f)) and use A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$. All time series are smoothed by averaging over two months of lagged daily data to reduce the appearance of noise.

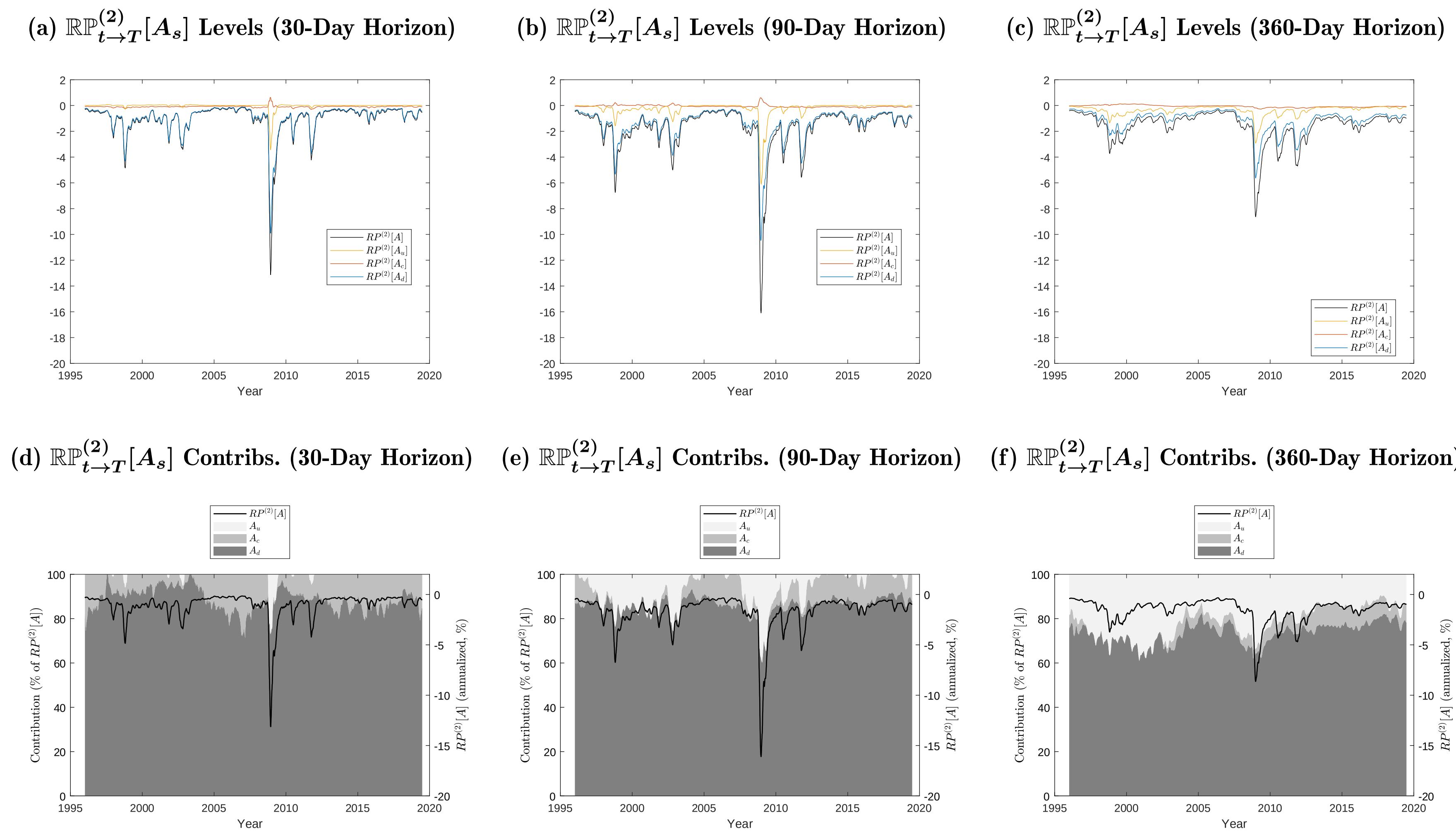


Figure 3

Data-Implied Variance Risk Premium Decomposition (Unrestricted Preference Parameters)

These graphs plot the data-implied unrestricted risk premium decompositions from Section 2.2 based on Proposition 3 with $n = 2$ (i.e., the variance risk premium). The decompositions use preference parameters reported in Table 1. Panels (a)-(c) plot the annualized risk premium levels at each date in percent. Risk premia are annualized by multiplying by each horizon (in units of fractions of a year). Panels (d)-(e) plot each component's contribution to the total risk premium at each date as a fraction of the total risk premium. The dark/medium/light shaded regions represent the downside/central/upside risk premium contributions, respectively, and are measured on the left vertical axes. The decompositions are computed at three horizons (30 days - Panels (a) and (d); 90 days - Panels (b) and (e); and 360 days - Panels (c) and (f)) and use A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$. All time series are smoothed by averaging over two months of lagged daily data to reduce the appearance of noise.

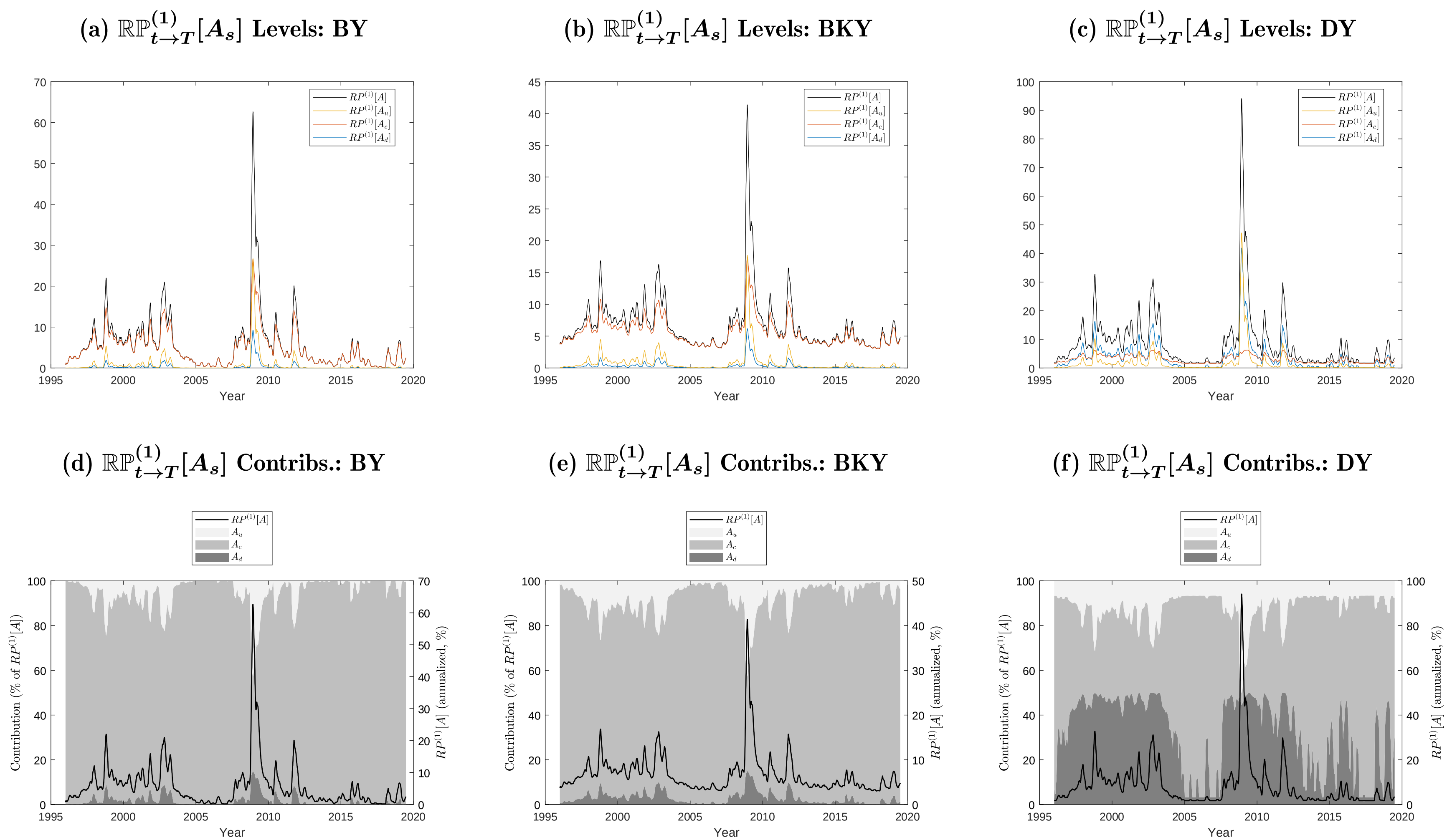
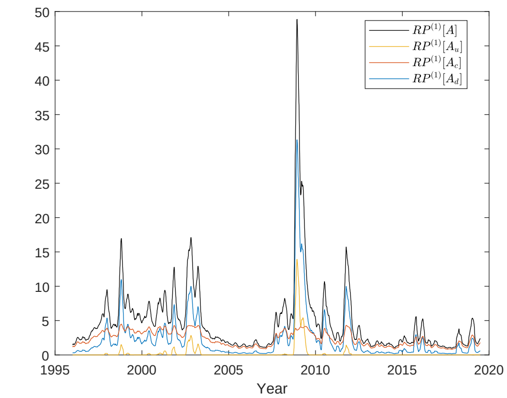


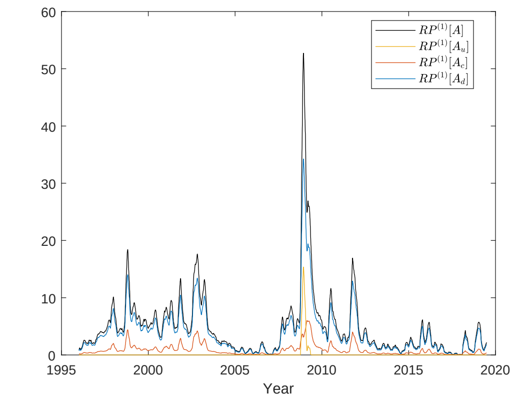
Figure 4
Model-Implied Market Risk Premium Decompositions (Long-Run Risk Models)

These graphs plot model-implied market risk premium decompositions based on Proposition 3 for the long-run risk models we study (see Subsections 3.1.1 and 3.1.2). Panels (a)/(d) plot results from the Bansal and Yaron (2004) model. Panels (b)/(e) plot results from the Bansal, Kiku, and Yaron (2012) model. Panels (c)/(f) plot results from the Drechsler and Yaron (2011) model. Panels (a)-(c) plot the annualized risk premium levels for each model at each date in percent. Panels (d)-(e) plot each component's contribution to the total risk premium for each model at each date as a fraction of the total risk premium. The dark/medium/light shaded regions represent the downside/central/upside risk premium contributions, respectively. All decompositions use a 30-day horizon to match model calibration frequencies in the original papers (monthly), and set A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$. All time series are smoothed by averaging over two months of lagged daily data to reduce the appearance of noise.

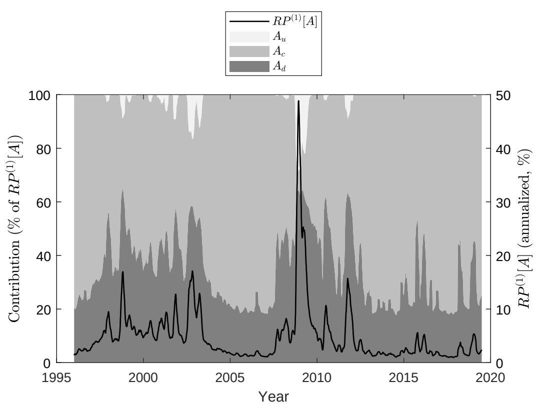
(a) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Levels: BEE (w/ Pref. Shocks)



(b) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Levels: BEE (w/o Pref. Shocks)



(c) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Contribs.: BEE (w/ Pref. Shocks)



(d) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Contribs.: BEE (w/o Pref. Shocks)

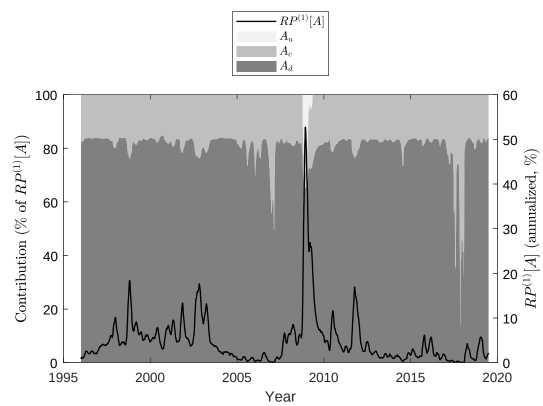


Figure 5

Model-Implied Market Risk Premium Decompositions (Habit Models)

These graphs plot model-implied market risk premium decompositions based on Proposition 3 for the habit formation models we study (see Subsection 3.2). Panels (a)/(c) plot results from the Bekaert, Engstrom, and Ermolov (2020) model with preference shocks. Panels (b)/(d) plot results from the Bekaert, Engstrom, and Ermolov (2020) model without preference shocks. Panels (a)-(b) plot the annualized risk premium levels for each model at each date in percent. Panels (c)-(d) plot each component's contribution to the total risk premium for each model at each date as a fraction of the total risk premium. The dark/medium/light shaded regions represent the downside/central/upside risk premium contributions, respectively. All decompositions use a 30-day horizon to match model calibration frequencies in the original papers (monthly), and set A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$. All time series are smoothed by averaging over two months of lagged daily data to reduce the appearance of noise.

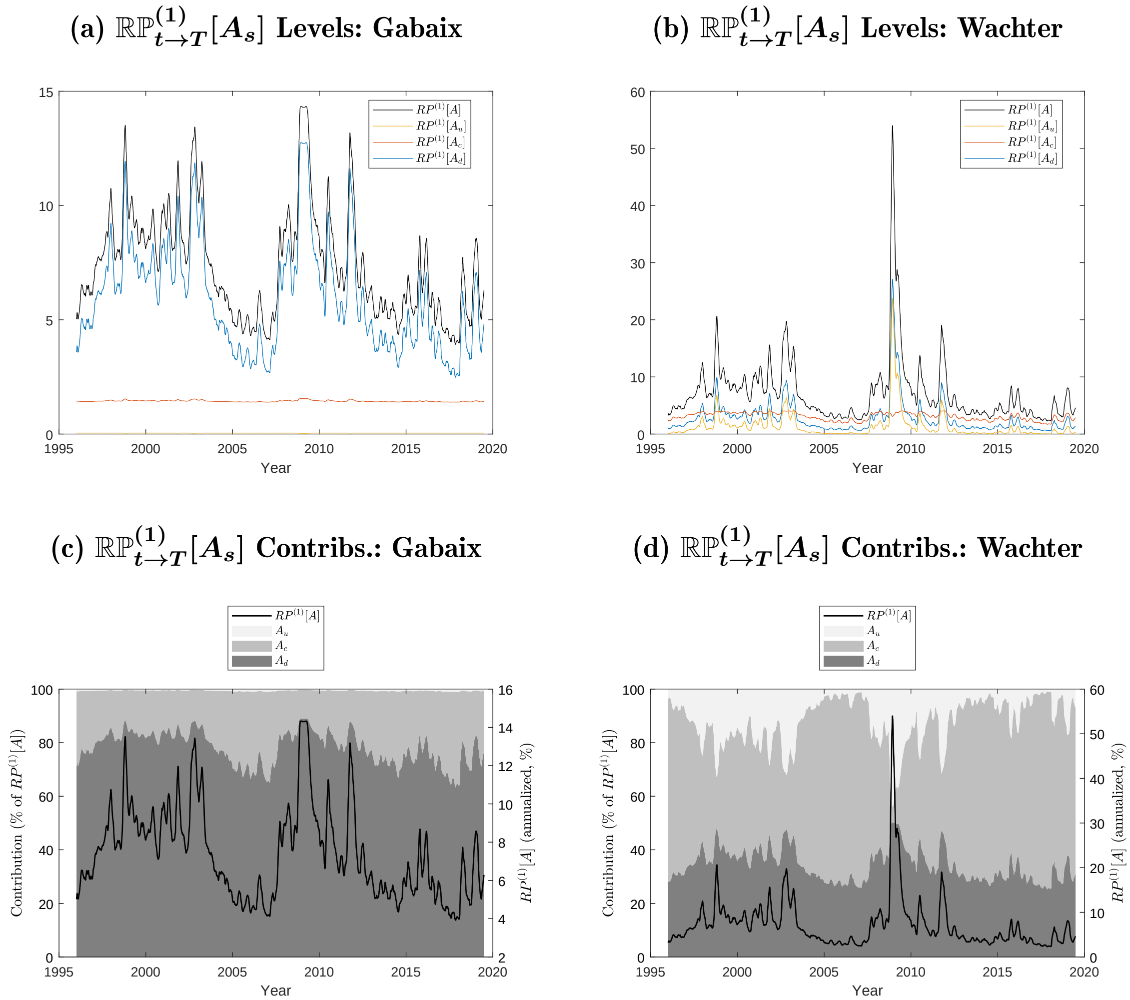
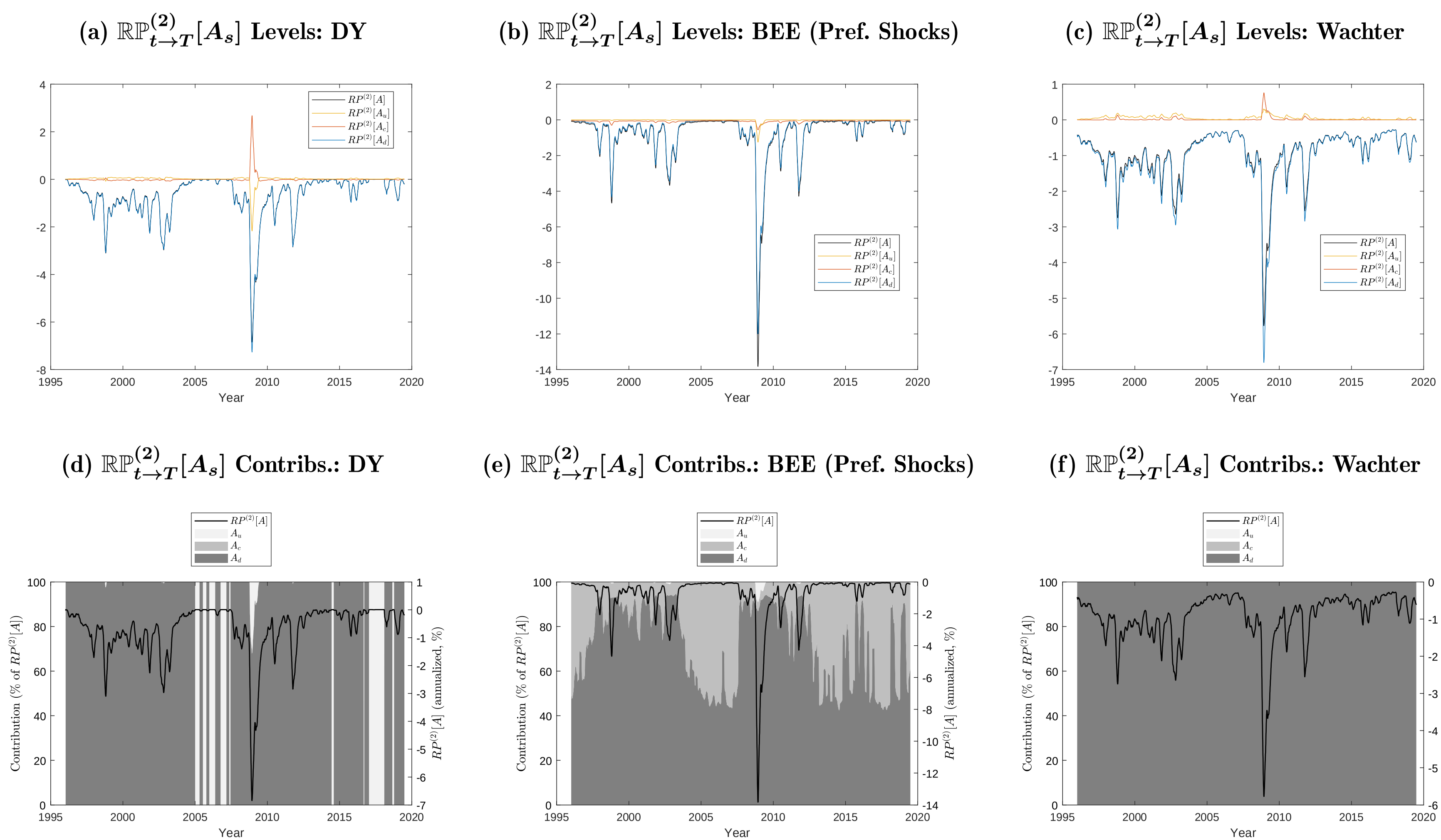


Figure 6

Model-Implied Market Risk Premium Decompositions (Disaster Models)

These graphs plot model-implied market risk premium decompositions based on Proposition 3 for the disaster models we study (see Subsection 3.3). Panels (a)/(c) plot results from the Gabaix (2012). Panels (b)/(d) plot results from the Wachter (2013) model. Panels (a)-(b) plot the annualized risk premium levels for each model at each date in percent. Panels (c)-(d) plot each component's contribution to the total risk premium for each model at each date as a fraction of the total risk premium. The dark/medium/light shaded regions represent the downside/central/upside risk premium contributions, respectively. All decompositions use a 30-day horizon to match model calibration frequencies in the original papers (monthly), and set A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$. All time series are smoothed by averaging over two months of lagged daily data to reduce the appearance of noise.



(a) $\mathbb{RP}^{(2)}_{t\to T}[A_s]$ Levels: DY

(b) $\mathbb{RP}^{(2)}_{t\to T}[A_s]$ Levels: BEE (Pref. Shocks)

(c) $\mathbb{RP}^{(2)}_{t\to T}[A_s]$ Levels: Wachter

(d) $\mathbb{RP}^{(2)}_{t\to T}[A_s]$ Contribs.: DY

(e) $\mathbb{RP}^{(2)}_{t\to T}[A_s]$ Contribs.: BEE (Pref. Shocks)

(f) $\mathbb{RP}^{(2)}_{t\to T}[A_s]$ Contribs.: Wachter

Figure 7

Model-Implied Variance Risk Premium Decompositions (Long-Run Risk Models)

These graphs plot model-implied variance risk premium decompositions based on Proposition 3 for the following models: Drechsler and Yaron (2011) (Panels (a)/(d)), Bekaert, Engstrom, and Ermolov (2020) with preference shocks (Panels (b)/(e)), and Wachter (2013) (Panels (c)/(f)). Note that the variance risk premium plot for the Gabaix (2012) model is similar, so we omit it for brevity. Panels (a)-(c) plot the annualized risk premium levels for each model at each date in percent. Risk premia are annualized by multiplying by each horizon (in units of fractions of a year). Panels (d)-(e) plot each component's contribution to the total risk premium for each model at each date as a fraction of the total risk premium. The dark/medium/light shaded regions represent the downside/central/upside risk premium contributions, respectively. All decompositions use a 30-day horizon to match model calibration frequencies in the original papers (monthly), and set $A_d{=}[0,0.9]$, $A_c{=}[0.9,1.1]$, and $A_u{=}[1.1,+\infty)$. All time series are smoothed by averaging over two months of lagged daily data to reduce the appearance of noise.

Table 1
Preference Parameter Estimates

This table reports preference parameters estimates for $\tau(x_s)$ (Equation 14), $\rho(x_s)$ (Equation 15), and $\kappa(x_s)$ (Equation 16) with $s \in \{d, c, u\}$ and $x_d = 0.85$, $x_c = 1.00$, and $x_u = 1.15$. These correspond to the three regions of interest in the gross market return space defined by A_d=[0, 0.9], A_c=[0.9, 1.1], and A_u=[1.1, +∞). $1/\tau(x_s)$ is the relative risk aversion. Parameters are estimated separately for each of five horizons (30, 60, 90, 180, and 360 days). Estimations are done using nonlinear least squares according to the description in Subsection 2.2. Values in brackets represent the 95% confidence intervals obtained from 10,000 block bootstrap simulations. The block length is set to be four years for simulations at all horizons. Parameters are estimated using daily S&P 500 excess market return data (ex dividend) from CRSP and risk-neutral moments computed from daily option prices obtained from Option Metrics. Data is daily and ranges from January, 1996 through June, 2019.

		Horizon (days)				
Region	**Parameter**	**30**	**60**	**90**	**180**	**360**
A_d	$1/\tau(x_d)$	3.54	3.98	3.38	2.41	1.03
		[1.76, 7.04]	[2.21, 6.77]	[1.78, 6.05]	[1.24, 6.79]	[0.71, 2.38]
	$\tau(x_d)$	0.28	0.25	0.30	0.42	0.97
		[0.14, 0.57]	[0.15, 0.45]	[0.17, 0.56]	[0.15, 0.81]	[0.42, 1.42]
	$\rho(x_d)$	0.99	1.03	1.05	1.20	1.92
		[0.81, 1.33]	[0.97, 1.34]	[0.96, 1.52]	[0.97, 1.99]	[1.02, 2.52]
	$\kappa(x_d)$	0.81	0.95	0.96	1.27	3.09
		[0.62, 1.38]	[0.89, 1.54]	[0.87, 1.98]	[0.92, 3.37]	[0.98, 5.19]
A_c	$1/\tau(x_c)$	2.04	2.19	2.01	1.60	0.85
		[1.13, 4.21]	[1.33, 3.29]	[1.16, 3.21]	[0.97, 3.44]	[0.70, 1.77]
	$\tau(x_c)$	0.49	0.46	0.50	0.63	1.18
		[0.24, 0.88]	[0.30, 0.75]	[0.31, 0.86]	[0.29, 1.03]	[0.56, 1.43]
	$\rho(x_c)$	1.82	1.75	1.74	1.86	2.29
		[0.76, 2.83]	[1.19, 2.67]	[1.00, 2.82]	[1.01, 3.01]	[0.79, 3.40]
	$\kappa(x_c)$	1.79	1.86	1.82	2.14	3.82
		[0.57, 2.69]	[1.20, 3.14]	[1.00, 3.62]	[1.08, 4.46]	[1.13, 5.14]
A_u	$1/\tau(x_u)$	1.04	1.01	0.97	0.95	0.71
		[0.75, 3.74]	[0.79, 1.86]	[0.74, 2.16]	[0.68, 2.09]	[0.59, 1.44]
	$\tau(x_u)$	0.96	0.99	1.03	1.05	1.41
		[0.27, 1.34]	[0.54, 1.26]	[0.46, 1.35]	[0.48, 1.46]	[0.69, 1.70]
	$\rho(x_u)$	5.76	7.10	6.82	4.91	3.00
		[0.65, 6.71]	[1.34, 8.76]	[0.77, 9.07]	[0.28, 8.44]	[0.00, 8.56]
	$\kappa(x_u)$	3.97	5.29	5.08	4.88	4.94
		[0.45, 4.66]	[2.00, 5.76]	[0.98, 6.25]	[1.18, 7.66]	[1.06, 7.81]

Table 2
Unrestricted Data-Implied Market Risk Premium Decomposition Summary Statistics

This table reports summary statistics for the unrestricted data-implied risk premium decomposition according to Proposition 3 using preference parameters reported in Table 1 with $n = 1$ (i.e., the market risk premium). Panel A reports statistics for the risk premium levels (annualized, in percent) and Panel B reports statistics for the contributions of risk premia from each region to the total risk premium (as fractions of the total risk premium, in percent). A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$ and these labels correspond to the downside, central, and upside risk premia, respectively. $A = A_d \cup A_c \cup A_u$ and this label corresponds to the total risk premium. Statistics reported under "Unconditional" use the full estimated time series for each risk premium measure. Statistics reported under "Cond. Means" report the means for each time series conditional on 30-day risk-neutral variance ($\mathbb{M}^{*(2)}_{t\to T}[A]$) falling below it's first quartile ("Lo"), between its first and third quartiles ("Mid"), or above its third quartile ("Hi"). These correspond to periods of low, moderate, or high market volatility, respectively. Statistics are reported for risk premium decompositions at 30, 60, 90, 180, and 360-day horizons, and are based on daily data from January, 1996 through June, 2019.

		Panel A: $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ (%)					Panel B: $\mathbb{RP}^{(1)}_{t\to T}[A_s]/\mathbb{RP}^{(1)}_{t\to T}[A]$ (%)				
		Cond. Means			Unconditional		Cond. Means			Unconditional	
Horizon (days)	**Region**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**
30	A	3.19	6.92	17.82	8.72	7.50					
	A_d	1.21	3.13	10.35	4.45	4.97	38.05	44.67	55.30	45.67	9.33
	A_c	1.88	3.37	4.71	3.33	1.18	59.84	50.43	30.67	47.84	13.32
	A_u	0.07	0.38	3.05	0.97	2.15	2.11	4.90	14.02	6.48	6.14
60	A	3.98	8.03	18.44	9.62	6.99					
	A_d	2.01	4.53	11.76	5.71	4.90	51.23	56.71	62.22	56.72	7.04
	A_c	1.73	2.47	2.68	2.34	0.46	45.38	32.87	16.73	31.96	12.05
	A_u	0.13	0.92	4.15	1.53	2.13	3.39	10.42	21.05	11.32	8.17
90	A	3.98	7.50	15.80	8.69	5.60					
	A_d	2.31	4.62	10.45	5.50	4.01	58.70	61.77	64.86	61.78	5.70
	A_c	1.36	1.62	1.55	1.54	0.19	35.53	23.34	11.05	23.32	10.16
	A_u	0.23	1.20	3.90	1.63	1.75	5.77	14.88	24.09	14.91	8.52
180	A	3.82	6.52	12.33	7.29	4.15					
	A_d	2.57	4.39	8.56	4.98	3.06	67.91	67.67	68.45	67.92	4.56
	A_c	0.68	0.64	0.58	0.64	0.09	18.54	10.89	5.17	11.37	5.63
	A_u	0.50	1.43	3.21	1.64	1.22	13.55	21.44	26.38	20.71	6.70
360	A	2.54	4.02	7.17	4.44	2.48					
	A_d	1.79	2.76	4.82	3.03	1.71	71.39	69.12	67.10	69.18	4.86
	A_c	0.19	0.19	0.20	0.19	0.05	8.08	5.20	2.95	5.36	2.45
	A_u	0.50	1.02	2.12	1.16	0.81	20.53	25.68	29.95	25.46	5.52

Table 3

Forecasting Regressions Using Unrestricted Data-Implied Risk Premia

This table reports excess market return forecasting regression results based on the specification in Equation 33 using risk premia from the unrestricted decomposition described in Subsection 2.2. Results under "Full Sample" use data from January, 1996 through June, 2019. Results under "Ex Crisis" use the same time period except they exclude data from August, 2008 through January, 2009 (i.e., data from the height of the 2008 Financial Crisis). Results are reported for 30, 90, and 360-day forecast horizons. Excess market returns are measured as ex dividend returns on the S&P 500 index obtained from CRSP less the risk-free rate obtained from Kenneth French's website. Data is daily and excess returns at each horizon are computed by compounding daily returns to the horizon of interest and subtracting the compounded risk-free rate. T-statistics are reported in parentheses and are computed according to Newey and West (1987) with lag values according to Newey and West (1994), with one slight modification. Since we have overlapping data, we multiply the Newey and West (1994)-implied lag value by the number of trading days in each horizon and use this as our lag value. R^2_{IS} indicates standard adjusted in-sample R-squared values. $R^2_{pseudoOS}$ corresponds to out-of-sample R-squared values computed according to the methodology in Goyal and Welch (2008) with two slight modifications. First, they are labeled as "pseudo" out-of-sample because they use preference parameters estimated using our full sample of data. Second, they require no estimation and just use our ex ante measure for the total risk premium as the model-implied market return forecast. Historical average returns are estimated using ex dividend S&P 500 excess returns starting in 1926 and obtained from CRSP. We do not report intercepts or $R^2_{pseudoOS}$ values for the truncated risk premia since they do not have the theoretical implications that $a_T = 0$ and $b_T = 1$ (as is the case for the total risk premium).

$$R_{M,t\to T} - R_{f,t\to T} = a_T + b_T \mathbb{RP}^{(1)}[A_s] + \varepsilon_{t\to T}$$

	Full Sample												Ex Crisis											
Horizon (days)	30				90				360				30				90				360			
	I	*II*	*III*	*IV*	*I*	*II*	*III*	*IV*	*I*	*II*	*III*	*IV*	*I*	*II*	*III*	*IV*	*I*	*II*	*III*	*IV*	*I*	*II*	*III*	*IV*
a_T	3.14	-	-	-	1.22	-	-	-	0.05	-	-	-	-5.31	-	-	-	-5.84	-	-	-	1.03	-	-	-
	(0.64)	-	-	-	(0.25)	-	-	-	(0.01)	-	-	-	(-1.34)	-	-	-	(-1.25)	-	-	-	(0.19)	-	-	-
$\mathbb{RP}^{(1)}_{t\to T}$	0.33				0.50				1.31				1.63				1.64				1.57			
	(0.51)				(0.77)				(2.01)				(3.87)				(3.39)				(1.80)			
$\mathbb{RP}^{(1)}_{t\to T}[A_d]$		0.30				0.51				1.46				1.32				1.61				1.70		
		(0.61)				(0.85)				(2.17)				(4.18)				(3.92)				(1.98)		
$\mathbb{RP}^{(1)}_{t\to T}[A_c]$			0.13				-0.99				8.05				1.34				-0.73				9.36	
			(0.11)				(-0.57)				(4.46)				(1.61)				(-0.43)				(3.86)	
$\mathbb{RP}^{(1)}_{t\to T}[A_u]$				0.10				0.20				0.67				0.75				0.75				0.75
				(0.39)				(0.52)				(0.99)				(4.77)				(2.34)				(0.78)
R^2_{IS}	0.00	0.00	0.00	0.00	0.01	0.01	0.00	0.00	0.04	0.05	0.25	0.01	0.03	0.03	0.01	0.04	0.08	0.09	0.00	0.05	0.05	0.06	0.33	0.02
$R^2_{pseudoOS}$	-0.02	-	-	-	-0.05	-	-	-	0.02	-	-	-	0.02	-	-	-	0.06	-	-	-	0.09	-	-	-

Table 4

Unrestricted Data-Implied Variance Risk Premium Decomposition Summary Statistics

This table reports summary statistics for the unrestricted data-implied risk premium decomposition according to Proposition 3 using preference parameters reported in Table 1 with $n = 2$ (i.e., the variance risk premium). Panel A reports statistics for the risk premium levels and Panel B reports statistics for the contributions of risk premia from each region to the total risk premium (as fractions of the total risk premium, in percent). Risk premia are annualized by multiplying by each horizon (in units of fractions of a year). A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$ and these labels correspond to the downside, central, and upside risk premia, respectively. $A = A_d \cup A_c \cup A_u$ and this label corresponds to the total risk premium. Statistics reported under "Unconditional" use the full estimated time series for each risk premium measure. Statistics reported under "Cond. Means" report the means for each time series conditional on 30-day risk-neutral variance ($\mathbb{M}^{*(2)}_{t\to T}[A]$) falling below it's first quartile ("Lo"), between its first and third quartiles ("Mid"), or above its third quartile ("Hi"). These correspond to periods of low, moderate, or high market volatility, respectively. Statistics are reported for risk premium decompositions at 30, 60, 90, 180, and 360-day horizons, and are based on daily data from January, 1996 through June, 2019.

		Panel A: $\mathbb{RP}^{(2)}_{t\to T}[A_s]$ (%)					**Panel B: $\mathbb{RP}^{(2)}_{t\to T}[A_s]/\mathbb{RP}^{(2)}_{t\to T}[A]$ (%)**				
		Cond. Means			**Unconditional**		**Cond. Means**			**Unconditional**	
Horizon (days)	**Region**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**
30	A	-0.29	-0.67	-2.50	-1.03	1.46					
	A_d	-0.25	-0.60	-2.22	-0.92	1.19	83.28	89.15	92.11	88.42	6.94
	A_c	-0.05	-0.09	-0.11	-0.09	0.10	19.19	14.56	7.63	13.99	6.87
	A_u	0.01	0.02	-0.14	-0.02	0.33	-2.47	-3.71	0.26	-2.41	4.54
60	A	-0.51	-1.10	-3.65	-1.59	1.99					
	A_d	-0.44	-0.97	-2.97	-1.34	1.46	87.83	89.79	85.96	88.34	4.05
	A_c	-0.07	-0.10	-0.02	-0.07	0.12	14.06	9.94	2.43	9.09	5.63
	A_u	0.01	-0.01	-0.61	-0.16	0.63	-1.89	0.27	11.62	2.57	6.77
90	A	-0.62	-1.26	-3.72	-1.71	1.91					
	A_d	-0.54	-1.08	-2.88	-1.39	1.32	88.44	87.61	81.84	86.37	4.38
	A_c	-0.07	-0.08	0.03	-0.05	0.10	12.56	7.56	-0.01	6.92	5.98
	A_u	0.01	-0.07	-0.83	-0.24	0.69	-1.00	4.83	18.17	6.71	8.60
180	A	-0.88	-1.63	-4.13	-2.07	2.03					
	A_d	-0.73	-1.31	-3.05	-1.60	1.39	85.57	82.37	76.88	81.80	4.35
	A_c	-0.08	-0.04	0.08	-0.02	0.11	8.99	2.95	-2.30	3.15	5.86
	A_u	-0.04	-0.25	-1.11	-0.41	0.71	5.44	14.68	25.42	15.06	9.04
360	A	-0.81	-1.34	-2.67	-1.54	1.21					
	A_d	-0.62	-0.98	-1.83	-1.10	0.82	78.33	74.97	69.77	74.51	4.61
	A_c	-0.06	-0.05	-0.05	-0.06	0.09	7.09	3.72	0.93	3.87	5.20
	A_u	-0.11	-0.28	-0.77	-0.36	0.39	14.58	21.31	29.30	21.62	7.41

Table 5

Restricted Data-Implied Market Risk Premium Decomposition Summary Statistics

This table reports summary statistics for the restricted data-implied risk premium decomposition according to Proposition 3 with $n = 1$ (i.e., the market risk premium). The decompositions set preference parameters to be $\tau = 1$, $\rho = 2$, and $\kappa = 4$ across all regions and horizons. Panel A reports statistics for the risk premium levels (annualized, in percent) and Panel B reports statistics for the contributions of risk premia from each region to the total risk premium (as fractions of the total risk premium, in percent). A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$ and these labels correspond to the downside, central, and upside risk premia, respectively. $A = A_d \cup A_c \cup A_u$ and this label corresponds to the total risk premium. Statistics reported under "Unconditional" use the full estimated time series for each risk premium measure. Statistics reported under "Cond. Means" report the means for each time series conditional on 30-day risk-neutral variance ($\mathbb{M}^{*(2)}_{t\to T}[A]$) falling below it's first quartile ("Lo"), between its first and third quartiles ("Mid"), or above its third quartile ("Hi"). These correspond to periods of low, moderate, or high market volatility, respectively. Statistics are reported for risk premium decompositions at 30, 60, 90, 180, and 360-day horizons, and are based on daily data from January, 1996 through June, 2019.

		Panel A: $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ (%)					**Panel B: $\mathbb{RP}^{(1)}_{t\to T}[A_s]/\mathbb{RP}^{(1)}_{t\to T}[A]$ (%)**				
		Cond. Means			**Unconditional**		**Cond. Means**			**Unconditional**	
Horizon (days)	**Region**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**
30	A	1.61	3.53	10.19	4.72	4.96					
	A_d	0.64	1.65	6.05	2.50	3.30	39.75	45.91	57.16	47.18	10.09
	A_c	0.93	1.68	2.36	1.66	0.60	58.20	49.26	29.12	46.46	13.74
	A_u	0.03	0.19	1.71	0.53	1.36	2.05	4.83	13.71	6.35	6.07
60	A	1.91	3.94	10.33	5.03	4.69					
	A_d	1.03	2.32	6.71	3.09	3.22	53.32	58.51	64.17	58.63	7.81
	A_c	0.81	1.16	1.25	1.09	0.22	43.42	31.38	15.18	30.34	12.16
	A_u	0.06	0.44	2.31	0.81	1.41	3.26	10.11	20.65	11.03	8.20
90	A	2.16	4.23	10.31	5.24	4.45					
	A_d	1.34	2.72	6.88	3.41	3.06	61.23	64.12	66.53	64.00	6.36
	A_c	0.70	0.83	0.79	0.79	0.10	33.25	21.46	9.46	21.40	10.01
	A_u	0.12	0.65	2.58	1.00	1.41	5.52	14.43	24.01	14.60	8.78
180	A	2.70	4.84	10.30	5.67	4.07					
	A_d	1.91	3.36	7.03	3.92	2.77	70.41	69.71	68.61	69.61	5.20
	A_c	0.44	0.42	0.37	0.41	0.06	17.05	9.74	4.23	10.19	5.43
	A_u	0.33	1.02	2.83	1.30	1.34	12.55	20.55	27.16	20.20	7.50
360	A	3.35	5.50	10.34	6.17	3.79					
	A_d	2.49	3.91	6.97	4.32	2.49	74.17	71.38	68.01	71.24	5.09
	A_c	0.24	0.24	0.25	0.24	0.06	7.51	4.80	2.62	4.93	2.31
	A_u	0.60	1.32	3.05	1.57	1.31	18.32	23.81	29.37	23.83	6.00

Table 6

Representative Agent Model-Implied Market Risk Premium Decomposition Summary Statistics

This table reports summary statistics for the model-implied risk premium decompositions based on representative agent models described in Section 3 with $n = 1$ (i.e., the market risk premium). Panel A reports statistics for the risk premium levels (annualized, in percent) and Panel B reports statistics for the contributions of risk premia from each region to the total risk premium (as fractions of the total risk premium, in percent). Results are reported for the following models: Bansal and Yaron (2004) ("BY"), Bansal, Kiku, and Yaron (2012) ("BKY"), Drechsler and Yaron (2011) ("DY"), Bekaert, Engstrom, and Ermolov (2020) ("BEE") with and without preference shocks, Gabaix (2012) ("Gabaix"), and Wachter (2013) ("Wachter"). A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$ and these labels correspond to the downside, central, and upside risk premia, respectively. $A = A_d \cup A_c \cup A_u$ and this label corresponds to the total risk premium. Statistics reported under "Unconditional" use the full estimated time series for each risk premium measure. Statistics reported under "Cond. Means" report the means for each time series conditional on 30-day risk-neutral variance ($\mathbb{M}^{*(2)}_{t\to T}[A]$) falling below it's first quartile ("Lo"), between its first and third quartiles ("Mid"), or above its third quartile ("Hi"). These correspond to periods of low, moderate, or high market volatility, respectively. Results are based on state variables extracted from the data under each model using their original calibrations, which are monthly in all cases, and use daily data from January, 1996 through June, 2019.

			Panel A: $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ (%)					Panel B: $\mathbb{RP}^{(1)}_{t\to T}[A_s]/\mathbb{RP}^{(1)}_{t\to T}[A]$ (%)				
			Cond. Means			Unconditional		Cond. Means			Unconditional	
Class	**Model**	**Region**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**
LRR	BY	A	0.63	3.84	14.33	5.66	7.61					
		A_d	0.00	0.03	1.04	0.28	0.98	0.00	0.47	4.98	1.48	2.66
		A_c	0.63	3.70	10.44	4.62	4.38	100.00	97.66	81.18	94.13	9.69
		A_u	0.00	0.11	2.84	0.76	2.73	0.00	1.86	13.84	4.39	7.05
	BKY	A	3.59	5.64	12.13	6.75	4.68					
		A_d	0.01	0.13	0.99	0.32	0.70	0.37	2.04	6.76	2.80	2.89
		A_c	3.53	5.17	8.59	5.61	2.27	98.41	92.30	76.04	89.76	10.06
		A_u	0.05	0.35	2.54	0.82	1.88	1.22	5.66	17.20	7.44	7.20
	DY	A	1.76	5.54	21.21	8.51	11.23					
		A_d	0.08	2.20	10.29	3.69	5.50	4.17	34.73	48.68	30.58	18.44
		A_c	1.56	2.78	5.13	3.06	1.46	89.20	56.25	30.30	58.00	23.54
		A_u	0.12	0.56	5.79	1.76	4.91	6.63	9.02	21.02	11.42	7.25

Table 6

Representative Agent Model-Implied Market Risk Premium Decomposition Summary Statistics (continued)

			Panel A: $\mathbb{RP}^{(1)}_{t\to T}[\boldsymbol{A}_s]$ (%)					Panel B: $\mathbb{RP}^{(1)}_{t\to T}[\boldsymbol{A}_s]/\mathbb{RP}^{(1)}_{t\to T}[\boldsymbol{A}]$ (%)				
			Cond. Means			Unconditional		Cond. Means			Unconditional	
Class	**Model**	**Region**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**
Habit	**BEE (w/ Pref. Shocks)**	$\boldsymbol{A}$	1.35	3.29	11.47	4.85	5.84					
		$\boldsymbol{A}_d$	0.27	1.11	6.46	2.23	3.80	19.47	30.77	52.66	33.42	13.94
		$\boldsymbol{A}_c$	1.09	2.18	3.83	2.32	1.12	80.53	69.23	42.11	65.28	16.79
		$\boldsymbol{A}_u$	0.00	0.00	1.18	0.30	1.47	0.00	0.00	5.23	1.31	4.43
	BEE (w/o Pref. Shocks)	$\boldsymbol{A}$	0.65	3.35	12.07	4.86	6.36					
		$\boldsymbol{A}_d$	0.52	2.79	9.27	3.84	4.48	68.94	83.47	79.53	78.85	14.31
		$\boldsymbol{A}_c$	0.13	0.56	2.26	0.88	1.09	31.06	16.53	19.33	20.86	14.31
		$\boldsymbol{A}_u$	0.00	0.00	0.55	0.14	1.40	0.00	0.00	1.14	0.29	2.65
Disaster	**Gabaix**	$\boldsymbol{A}$	4.72	7.06	10.82	7.42	2.50					
		$\boldsymbol{A}_d$	3.28	5.58	9.29	5.93	2.47	69.07	78.61	85.48	77.94	6.52
		$\boldsymbol{A}_c$	1.40	1.43	1.49	1.44	0.04	30.02	20.76	14.12	21.42	6.31
		$\boldsymbol{A}_u$	0.04	0.04	0.04	0.04	0.00	0.91	0.63	0.39	0.64	0.20
	Wachter	$\boldsymbol{A}$	3.04	5.73	14.31	7.20	6.24					
		$\boldsymbol{A}_d$	0.83	1.93	6.36	2.76	3.21	27.26	32.91	42.48	33.89	6.18
		$\boldsymbol{A}_c$	2.13	3.17	3.92	3.10	0.72	70.37	57.20	33.14	54.48	15.33
		$\boldsymbol{A}_u$	0.08	0.63	4.03	1.34	2.67	2.37	9.89	24.38	11.63	9.17

Table 7
Representative Agent Model-Implied Variance Risk Premium Decomposition Summary Statistics

This table reports summary statistics for the model-implied risk premium decompositions based on representative agent models described in Section 3 with $n = 2$ (i.e., the variance risk premium). Panel A reports statistics for the risk premium levels (annualized by multiplying by 12, in percent) and Panel B reports statistics for the contributions of risk premia from each region to the total risk premium (as fractions of the total risk premium, in percent). Results are reported for the following models: Drechsler and Yaron (2011) ("DY"), Bekaert, Engstrom, and Ermolov (2020) ("BEE") with and without preference shocks, Gabaix (2012) ("Gabaix"), and Wachter (2013) ("Wachter"). A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$ and these labels correspond to the downside, central, and upside risk premia, respectively. $A = A_d \cup A_c \cup A_u$ and this label corresponds to the total risk premium. Statistics reported under "Unconditional" use the full estimated time series for each risk premium measure. Statistics reported under "Cond. Means" report the means for each time series conditional on 30-day risk-neutral variance ($\mathbb{M}^{*(2)}_{t\to T}[A]$) falling below it's first quartile ("Lo"), between its first and third quartiles ("Mid"), or above its third quartile ("Hi"). These correspond to periods of low, moderate, or high market volatility, respectively. Results are based on state variables extracted from the data under each model using their original calibrations, which are monthly in all cases, and use daily data from January, 1996 through June, 2019.

			Panel A: $\mathbb{RP}^{(2)}_{t\to T}[\boldsymbol{A_s}]$ (%)					Panel B: $\mathbb{RP}^{(2)}_{t\to T}[\boldsymbol{A_s}]/\mathbb{RP}^{(2)}_{t\to T}[\boldsymbol{A}]$ (%)				
			Cond. Means			Unconditional		Cond. Means			Unconditional	
Class	**Model**	**Region**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**
LRR	**DY**	$\boldsymbol{A}$	0.00	-0.42	-1.94	-0.69	0.99					
		$\boldsymbol{A_d}$	-0.01	-0.44	-1.96	-0.71	1.01	-62.36	100.17	101.62	59.90	209.08
		$\boldsymbol{A_c}$	0.00	-0.01	0.06	0.01	0.25	4.09	1.83	1.26	2.25	4.87
		$\boldsymbol{A_u}$	0.01	0.03	-0.03	0.01	0.21	158.28	-2.00	-2.88	37.85	206.57
Habit	**BEE (w/ Pref. Shocks)**	$\boldsymbol{A}$	-0.07	-0.31	-2.46	-0.79	1.62					
		$\boldsymbol{A_d}$	-0.03	-0.25	-2.27	-0.70	1.46	46.26	70.41	92.51	69.90	19.57
		$\boldsymbol{A_c}$	-0.04	-0.06	-0.14	-0.07	0.07	53.75	29.65	7.11	30.04	19.68
		$\boldsymbol{A_u}$	0.00	0.00	-0.05	-0.01	0.12	0.00	-0.06	0.38	0.06	0.96
	BEE (w/o Pref. Shocks)	$\boldsymbol{A}$	-0.17	-0.91	-3.19	-1.30	1.60					
		$\boldsymbol{A_d}$	-0.16	-0.86	-2.74	-1.15	1.28	86.96	94.28	88.78	91.08	15.13
		$\boldsymbol{A_c}$	-0.01	-0.06	-0.34	-0.12	0.19	13.04	5.74	10.55	8.77	15.07
		$\boldsymbol{A_u}$	0.00	0.00	-0.10	-0.02	0.27	0.00	-0.02	0.67	0.16	2.22
Disaster	**Gabaix**	$\boldsymbol{A}$	-0.87	-2.46	-6.81	-3.15	2.68					
		$\boldsymbol{A_d}$	-0.87	-2.45	-6.74	-3.13	2.65	101.16	99.67	98.96	99.86	0.92
		$\boldsymbol{A_c}$	0.01	-0.01	-0.08	-0.03	0.04	-0.90	0.40	1.14	0.26	0.84
		$\boldsymbol{A_u}$	0.00	0.00	0.01	0.00	0.01	-0.26	-0.06	-0.11	-0.12	0.11
	Wachter	$\boldsymbol{A}$	-0.38	-0.78	-1.90	-0.96	0.76					
		$\boldsymbol{A_d}$	-0.39	-0.82	-2.10	-1.03	0.87	102.14	104.70	109.47	105.25	3.07
		$\boldsymbol{A_c}$	0.00	0.00	0.07	0.02	0.08	-0.44	0.19	-2.11	-0.54	1.67
		$\boldsymbol{A_u}$	0.01	0.04	0.13	0.06	0.06	-1.70	-4.89	-7.36	-4.71	2.33

Table 8
Average Conditional Differences Between Data- and Model-Implied Decompositions

This table reports summary statistics for the conditional differences between the unrestricted data-implied market (variance) risk premium decomposition time series (30-day horizon) summarized in Table 2 (4) and the corresponding model-implied risk premium decomposition time series based on representative agent models summarized in Table 8 (7). Panels A and C report statistics for the annualized level differences (i.e., $\mathbb{RP}^{(n)}_{t\to T,\,data}[A_s] - \mathbb{RP}^{(n)}_{t\to T,\,model}[A_s]$) for $n=1$ and $n=2$, respectively. Panels B and D report statistics for the contribution differences (i.e., $\mathbb{RP}^{(n)}_{t\to T,\,data}[A_s]/\mathbb{RP}^{(n)}_{t\to T,\,data}[A] - \mathbb{RP}^{(n)}_{t\to T,\,model}[A_s]/\mathbb{RP}^{(n)}_{t\to T,\,model}[A]$) for $n=1$ and $n=2$, respectively. Results are reported for the following models: Bansal and Yaron (2004) ("BY"), Bansal, Kiku, and Yaron (2012) ("BKY"), Drechsler and Yaron (2011) ("DY"), Bekaert, Engstrom, and Ermolov (2020) ("BEE") with and without preference shocks, Gabaix (2012) ("Gabaix"), and Wachter (2013) ("Wachter"). A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$ and these labels correspond to the downside, central, and upside risk premia, respectively. $A = A_d \cup A_c \cup A_u$ and this label corresponds to the total risk premium. T-statistics are reported in parentheses and are computed according to Newey and West (1987) with lag values according to Newey and West (1994), with one slight modification. Since we have overlapping data, we multiply the Newey and West (1994)-implied lag value by the number of overlapping days (21) and use this as our lag value. Data is daily and runs from January, 1996 through June, 2019.

		Panel A: $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ differences (%)				Panel B: $\mathbb{RP}^{(1)}_{t\to T}[A_s]/\mathbb{RP}^{(1)}_{t\to T}[A]$ differences (%)			Panel C: $\mathbb{RP}^{(2)}_{t\to T}[A_s]$ differences (%)				Panel D: $\mathbb{RP}^{(2)}_{t\to T}[A_s]/\mathbb{RP}^{(2)}_{t\to T}[A]$ differences (%)		
Class	**Model**	A	A_d	A_c	A_u	A_d	A_c	A_u	A	A_d	A_c	A_u	A_d	A_c	A_u
LRR	**BY**	3.06	4.18	-1.29	0.21	44.19	-46.28	2.09							
		(31.28)	(5.92)	(-2.08)	(2.26)	(33.34)	(-38.62)	(4.43)							
	BKY	1.97	4.14	-2.28	0.15	42.87	-41.92	-0.95							
		(3.68)	(5.74)	(-10.69)	(1.91)	(34.41)	(-44.12)	(-2.05)							
	DY	0.20	0.76	0.26	-0.79	15.10	-10.16	-4.94	-0.34	-0.20	-0.10	-0.03	28.52	11.74	-40.26
		(0.37)	(4.85)	(3.42)	(-2.26)	(4.79)	(-3.31)	(-9.83)	(-5.69)	(-5.71)	(-4.89)	(-2.51)	(2.03)	(10.71)	(-2.74)
Habit	**BEE (w/ Pref. Shocks)**	3.87	2.22	1.01	0.67	12.26	-17.43	5.18	-0.25	-0.22	-0.01	-0.01	18.52	-16.05	-2.47
		(10.25)	(9.19)	(17.17)	(3.96)	(6.74)	(-13.99)	(6.17)	(-4.67)	(-3.47)	(-1.03)	(-0.47)	(5.44)	(-4.85)	(-4.88)
	BEE (w/o Pref. Shocks)	3.86	0.61	2.45	0.83	-33.18	26.98	6.20	0.26	0.24	0.03	0.00	-2.65	5.22	-2.57
		(14.31)	(6.87)	(21.00)	(3.58)	(-14.95)	(10.67)	(5.40)	(3.05)	(3.56)	(1.01)	(0.11)	(-1.31)	(2.51)	(-5.52)
Disaster	**Gabaix**	1.30	-1.48	1.89	0.93	-32.27	26.43	5.84	2.12	2.21	-0.06	-0.03	-11.44	13.73	-2.29
		(1.55)	(-3.91)	(7.51)	(2.85)	(-28.36)	(19.21)	(4.53)	(6.52)	(6.59)	(-5.75)	(-0.80)	(-8.25)	(10.37)	(-3.79)
	Wachter	1.52	1.69	0.23	-0.37	11.78	-6.63	-5.15	-0.07	0.11	-0.10	-0.08	-16.83	14.53	2.30
		(5.36)	(5.56)	(2.24)	(-3.97)	(11.08)	(-5.67)	(-6.10)	(-0.86)	(2.74)	(-10.27)	(-2.00)	(-18.95)	(13.65)	(2.91)

Appendix A Proofs

Proof for the Coefficients in the Taylor expansion in Equation 13. Derivatives of $f_{x_s}(x)$ are given by:

$$
\begin{aligned}
\frac{\partial f_{x_s}(x)}{\partial x} &= -W_t U^{'}[W_t x_s] \frac{U^{''}[W_t x]}{(U^{'}[W_t x])^2}, \\
\frac{\partial^2 f_{x_s}(x)}{\partial^2 x} &= -W_t^2 U^{'}(W_t x_s) \left\{ \frac{U^{'''}(W_t x)}{(U^{'}(W_t x))^2} - 2\frac{\left(U^{''}(W_t x)\right)^2}{(U^{'}(W_t x))^3} \right\}, \text{and} \\
\frac{\partial^3 f_{x_s}(x)}{\partial^3 x} &= -W_t^3 U^{'}(W_t x_s) \left\{ \frac{U^{''''}(W_t x)}{(U^{'}(W_t x))^2} - 6\frac{U^{'''}(W_t x) u^{''}(W_t x)}{(U^{'}(W_t x))^3} + 6\frac{\left(U^{''}(W_t x)\right)^3}{(U^{'}(W_t x))^4} \right\}.
\end{aligned}
$$

Evaluating these derivatives at x_s and substituting in the definitions of $\tau(x_s)$, $\rho(x_s)$, and $\kappa(x_s)$ in Equations 14, 15, and 16 gives the desired result. ∎

Proof of Proposition 1. Assuming no-arbitrage, we can write

$$\mathbb{M}^{(n)}_{t\to T}[A_s] \equiv \mathbb{E}_t\left[\frac{M_{t\to T}}{\mathbb{E}_t[M_{t\to T}]}\frac{\mathbb{E}_t[M_{t\to T}]}{M_{t\to T}}(R_{M,t\to T} - R_{f,t\to T})^n \mathbb{I}_{A_s}\right] = \mathbb{E}^*_t\left[\frac{\mathbb{E}_t[M_{t\to T}]}{M_{t\to T}}(R_{M,t\to T} - R_{f,t\to T})^n \mathbb{I}_{A_s}\right] \quad \text{(A.1)}$$

Replacing $\mathbb{E}_t[M_{t\to T}]/M_{t\to T}$ in Equation A.1 with the expression in Equation 18 yields:

$$\mathbb{M}^{(n)}_{t\to T}[A_s] = \mathbb{E}^*_t\left[g_{x_s}(R_{M,t\to T})(R_{M,t\to T} - R_{f,t\to T})^n \mathbb{I}_{A_s}\right]. \quad \text{(A.2)}$$

Next, we replace the function $g_{x_s}(R_{M,t\to T})$in Equation A.2 with the expressions in Equations 10 and 11 and apply the definition of the covariance operator ($\mathbb{COV}^*_t[x,y] \equiv \mathbb{E}^*_t[xy] - \mathbb{E}^*_t[x]\,\mathbb{E}^*_t[y]$) to obtain

$$\mathbb{M}^{(n)}_{t\to T}[A_s] = \frac{\sum_{k=1}^{\infty}\theta_k(x_s)\,\mathbb{COV}^*_t\left[(R_{M,t\to T} - x_s)^k, (R_{M,t\to T} - R_{f,t\to T})^n \mathbb{I}_{A_s}\right]}{1 + \sum_{k=1}^{\infty}\theta_k(x_s)\,\mathbb{E}^*_t\left[(R_{M,t\to T} - x_s)^k\right]} + \mathbb{E}^*_t\left[(R_{M,t\to T} - R_{f,t\to T})^n \mathbb{I}_{A_s}\right].$$

Using our definition of $\mathbb{M}^{*(n)}_{t\to T}[A_s]$ and rearranging yields

$$\mathbb{M}^{(n)}_{t\to T}[A_s] - \mathbb{M}^{*(n)}_{t\to T}[A_s] = \frac{\sum_{k=1}^{\infty}\theta_k(x_s)\,\mathbb{COV}^*_t\left[(R_{M,t\to T} - x_s)^k, (R_{M,t\to T} - R_{f,t\to T})^n \mathbb{I}_{A_s}\right]}{1 + \sum_{k=1}^{\infty}\theta_k(x_s)\,\mathbb{E}^*_t\left[(R_{M,t\to T} - x_s)^k\right]}. \quad \text{(A.3)}$$

∎

Proof of Corollary 1. Replacing $(R_{M,t\to T}-x_s)^k$ in Equation A.3 with the expression in Equation 17 yields

$$\mathbb{M}^{(n)}_{t\to T}[A_s]-\mathbb{M}^{*(n)}_{t\to T}[A_s]=\frac{\sum_{k=1}^{\infty}\sum_{j=0}^{k}\lambda_t(x_s,k,j)\,\mathbb{COV}^*_t\left[(R_{M,t\to T}-R_{f,t\to T})^{k-j},(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_s}\right]}{1+\sum_{k=1}^{\infty}\sum_{j=0}^{k}\lambda_t(x_s,k,j)\,\mathbb{E}^*_t\left[(R_{M,t\to T}-R_{f,t\to T})^{k-j}\right]} \tag{A.4}$$

where $\lambda_t(x_s,k,j)$ is as defined in Equation 26. Applying the definition of the covariance operator ($\mathbb{COV}^*_t[x,y]\equiv\mathbb{E}^*_t[xy]-\mathbb{E}^*_t[x]\,\mathbb{E}^*_t[y]$), Equation A.4 simplifies to the result. ■

Proof of Proposition 3. We only consider the case where $n>1$ (the case where $n=1$ is straightforward). In this case, we start with the identity

$$(R_{M,t\to T}-\mathbb{E}_tR_{M,t\to T})^n=\sum_{s\in\{d,c,u\}}(R_{M,t\to T}-\mathbb{E}_tR_{M,t\to T})^n\,\mathbb{I}_{A_s}$$

and take expectations under the physical and risk-neutral measures to show

$$\begin{aligned}\mathbb{RP}^{(n)}_{t\to T}&=\sum_{s\in\{d,c,u\}}\mathbb{E}_t\left[(R_{M,t\to T}-\mathbb{E}_tR_{M,t\to T})^n\,\mathbb{I}_{A_s}\right]-\sum_{s\in\{d,c,u\}}\mathbb{E}^*_t\left[(R_{M,t\to T}-\mathbb{E}^*_tR_{M,t\to T})^n\,\mathbb{I}_{A_s}\right],\\&\equiv\sum_{s\in\{d,c,u\}}\mathbb{RP}^{(n)}_{t\to T}[A_s],\end{aligned}$$

where the second equality follows from our risk premium definition in Equation 29. Next, we can rewrite our expression for the risk premium when $n>1$ in Equation 29 as

$$\begin{aligned}\mathbb{RP}^{(n)}_{t\to T}[A_s]&=\mathbb{E}_t\left[(R_{M,t\to T}-R_{f,t\to T}+R_{f,t\to T}-\mathbb{E}_tR_{M,t\to T})^n\,\mathbb{I}_{A_s}\right]-\mathbb{M}^{*(n)}_{t\to T}[A_s]\\&=\mathbb{E}_t\left[\sum_{k=0}^{n}\frac{n!\,(-1)^{n-k}}{(n-k)!k!}\left(\mathbb{E}_t\left[R_{M,t\to T}-R_{f,t\to T}\right]\right)^{n-k}(R_{M,t\to T}-R_{f,t\to T})^k\,\mathbb{I}_{A_s}\right]-\mathbb{M}^{*(n)}_{t\to T}[A_s]\\&=\sum_{k=0}^{n}\frac{n!\,(-1)^{n-k}}{(n-k)!k!}\left(\mathbb{M}^{(1)}_{t\to T}[A]\right)^{n-k}\mathbb{M}^{(k)}_{t\to T}[A_s]-\mathbb{M}^{*(n)}_{t\to T}[A_s],\end{aligned}$$

where the second line follows from the Binomial theorem and the last time from our definition of $\mathbb{M}^{(n)}_{t\to T}[A_s]$. ■

Internet Appendix

"A Decomposition of Conditional Risk Premia and Implications for Representative Agent Models"

By Fousseni Chabi-Yo and Johnathan Loudis

The Internet Appendix is organized as indicated in the Table of Contents below. All figures and tables can be found at the end of the end of the appendix.

Contents

IA.1 Notation

Symbol	Description
t	Generic current date
T	Generic future date
S_t	Market index value at date t
$R_{M,t\to T}$	Gross market return from date t to T (Note: $R_{M,t\to T} = S_T/S_t$)
$R_{f,t\to T}$	Risk-free rate from date t to T
$C_{t\to T}[K]$	Call price at date t with expiration at date T and strike price K
$P_{t\to T}[K]$	Put price at date t with expiration at date T and strike price K
*	Denotes the risk-neutral measure; for instance, $\mathbb{VAR}^*$ denotes risk-neutral variance.
$\mathbb{E}_t^*[\cdot]$	Expectation at time t under the risk-neutral measure
$\mathbb{E}_t[\cdot]$	Expectation at time t under the physical measure
$Prob_t^*[B]$	Risk-neutral probability at time t of event B
A_s	Set describing regions of the gross market return space with $s \in \{d, c, u\}$ (see Equations 4, 5, and 6); note that $A \equiv A_d \cup A_c \cup A_u$
$\mathbb{I}_{A_s}$	Indicator function for gross market return inclusion in set A_s
$\underline{x}$	Threshold in gross market return space for computing downside truncated moments (see Equation 4); we set $\underline{x} = 0.9$ for all empirical results
$\overline{x}$	Threshold in gross market return space for computing upside truncated moments (see Equation 6); we set $\overline{x} = 1.1$ for all empirical results
x_s	Points in gross market return space around which Taylor expansions are taken and at which preference parameters are estimated; $s \in \{d, c, u\}$ (see Equations 4, 5, and 6); we set $x_d = 0.85$, $x_c = 1$, and $x_u = 1.15$ for all empirical results
$\tau(x_s)$	Risk tolerance evaluated at x_s (see Equation 14)
$\rho(x_s)$	Skewness tolerance evaluated at x_s (see Equation 15)
$\kappa(x_s)$	Kurtosis tolerance evaluated at x_s (see Equation 16)
$\theta_k(x_s)$	Parameters related to utility function derivatives (see Equation 12)
$\lambda_t(x_s, k, j)$	Function of preference parameters defined in Equation 26
$\mathbb{M}_{i,t\to T}^{*(n)}[A_s]$	Shorthand for $\mathbb{E}_t^*\left[(R_{i,t\to T} - R_{f,t\to T})^n \mathbb{I}_{A_s}\right]$ (see Equation 20)
$\mathbb{M}_{i,t\to T}^{(n)}[A_s]$	Shorthand for $\mathbb{E}_t\left[(R_{i,t\to T} - R_{f,t\to T})^n \mathbb{I}_{A_s}\right]$ (see Equation 21)
$\mathbb{RP}_{t\to T}^{(n)}[A_s]$	Shorthand for the risk premia defined in Equation 29
$U(W_T)$	Generic representative investor utility function evaluated at investor wealth, W_T
$M_{t\to T}$	Stochastic discount factor realization over the time period from t to T

IA.2 Robustness and Additional Results

In this section, we explore the robustness of our results to various modifications and provide some additional results. In Subsection IA.2.1, we provide plots of the preference parameters reported in Table 1 and discuss their implications for the conditional SDF. In Subsection IA.2.2, we provide market risk premium decomposition plots under the restricted preference parameter assumptions from Subsection 2.3. In Subsection IA.2.3, we construct all risk-neutral moments from observed options prices rather than using the implied volatility curve fitting technique (see Footnote 13) and show this does not materially alter our main results. In Subsection IA.2.4, we explore the effect of potential option mispricing due to supply/demand imbalances and show that this does not materially alter our main results.

IA.2.1 Unrestricted Preference Parameter Plots and the Implied SDF

Figure IA.1 plots estimates relative risk aversion and Figure IA.2 plots estimates of τ, ρ, and κ reported in Table 1 for visualization. These estimated preference parameters have implications for the behavior of the SDF in different regions of the return space. Given a set of estimated preference parameters in each region and measures of risk neutral moments at date t for horizon T, we can construct the conditional implied SDF in each region as the inverse of Equation 18. We do this for three different dates in Figure IA.3. The dates are chosen to be on dates with low market volatility (3 January, 2006 and 2 January, 2014) and one date with large market volatility (15 September, 2008). Within a given horizon, the SDF plots are very consistent across dates implying that the SDF does not change much over time.[38] It should also be noted that, although different preference parameters are used to construct the implied SDF in different regions, the SDF does not jump much at the boundaries between different regions relative to the overall range of the SDF across the plotted market return space. The SDF flattens with horizon, implying investor marginal utility increases more for equivalent decreases in market value in the short term than the long term.

IA.2.2 Restricted Preference Parameter Decomposition (Additional Results)

We provide market risk premium decomposition plots under the restricted preference parameter assumptions from Subsection 2.3 in Figure IA.4. We provide these plots here to save space in the main draft and since they are similar in appearance to those from our main unrestricted preference parameter results provided in Figure 2.

[38] The only exception is during November, 2008 in the peak of the financial crisis. During this period, the upper limit of the SDF became lower in the down region of the return space. This implies that during the financial crises investor marginal utility was particularly high in states of the world with very low realized market returns.

Table IA.1 provides forecasting regression results of the form in Equation 33 but using the risk premium decomposition constructed using the restricted preference parameters. In this case, we do not necessarily expect the null hypotheses that $a_T = 0$ and $b_T = 1$ to be satisfied to the extent that the total risk premium computed here represents a lower bound, except in the case that this lower bound is tight. In the table, we see patterns similar to those in Table 3. In fact, we cannot reject the separate nulls that $a_T = 0$ at any horizon whether we use the Full Sample or Ex Crisis samples. However, we can reject the null that $b_T = 1$ at the 30- and 90-day horizons when using the Ex Crisis sample. The fact that $b_T > 1$ is consistent with this measure representing a lower bound on the risk premium. All of the out-of-sample R-squared values are slightly negative when using the Full Sample. However, the out-of-sample R-squared values become positive in the Ex Crisis period with magnitudes similar to those using risk premia computed from estimated preference parameters.

IA.2.3 Observed Prices Instead of Implied Volatility Fitting

In this subsection, we explore whether constructing risk-neutral moments by numerically integrating over observed options prices rather than using the implied volatility fitting methodology alters our main results related to the market risk premium decomposition. Aside from numerically integrating over observed option prices (using the same equations summarized in Internet Appendix IA.4) rather than prices imputed from fitted implied volatility curves, all procedures are the same. This includes re-estimating preference parameters, which are similar to the original estimates so we do not report these. Results for the modified market risk premium decomposition are provided in Figure IA.5 and Table IA.2. These results can be compared with our analogous main results in Figure 2 and Table 2. The modified decomposition produces results that are both qualitatively and quantitatively similar to our main results and we conclude that the choice to compute risk-neutral moments by integrating over options prices implied by fitted implied volatility curves versus integrating over observed prices is innocuous.

IA.2.4 Potentially Overpriced Options

There is evidence that the well-known implied volatility smirk displayed by index option prices is at least partially the result of supply/demand imbalances caused by the inability of market makers to perfectly hedge option exposures (Garleanu, Pedersen, and Poteshman, 2009). These imbalances imply that observed option prices are mispriced and do not reflect no-arbitrage prices. Garleanu, Pedersen, and Poteshman (2009) define mispricing in the cross-section of option moneyness as deviations in measured Black-Scholes implied volatility from the physical volatility estimated using Bates (2006). The authors show that this difference is on average decreasing in option moneyness (see Figure 1 in their paper). Importantly, they show that (on average) the implied volatility of OTM calls is approximately the same as the

Bates (2006) volatility measure (i.e., these options are "correctly" priced if the Bates (2006) measure is the correct measure of volatility).

If the option prices we observed deviate from their no-arbitrage prices due to demand pressure, this is a problem for the risk-neutral moments we construct and use to perform the empirical implementation of our risk premium decomposition. Although a full estimation of demand-driven option pricing models and their effects on observed option prices is beyond the scope of this paper, we construct a heuristic method for assessing the potential effect of such demand pressure on observed options prices. We do this by modifying our measured implied volatility curves for cross-sections of option prices (in moneyness) at each date and for each maturity. Our main task is to provide a reasonable transformation of observed implied volatility curves to account for potential demand-driven mispricing and effectively correct observed options prices so that they reflect no-arbitrage relationships.

One approach would be to simply use the Bates (2006) volatility measure as the "correct" measure of implied volatility across all strikes for each date/maturity combination. We feel that this is too restrictive since it implies index prices are log-normal (i.e., the standard Black-Scholes assumption). There is significant evidence that this is not the case and that index prices are negatively skewed. To allow for this empirical fact, we consider approaches to shrink observed implied volatility curves towards the constant Black-Scholes benchmark. Two key insights from Garleanu, Pedersen, and Poteshman (2009) are that demand-corrected implied volatility curves should be lower than observed implied volatility curves and that OTM calls are approximately correctly priced. Given these insights, we assume that the OTM call with the lowest observed implied volatility is approximately correctly priced for sets of options at each date/maturity. We denote the associated implied volatility and strike price as IV_0 and K_0, respectively. We then construct a transformed implied volatility curve for each date/maturity as follows. Let all observed volatilities and strikes be denoted by IV_i and K_i. Let the transformed implied volatilities be denoted by $\tilde{IV}_i$. For any OTM calls with $K_i > K_0$, we set $\tilde{IV}_i = IV_0$. Next, for any options with $IV_i < IV_0$, we set $\tilde{IV}_i = IV_0$. Finally, we select a constant, a, and transform all other implied volatilities according to: $\tilde{IV}_i = IV_i - a\,(IV_i - IV_0)$. As long as $a \in [0, 1]$, this transformation shrinks the observed implied volatilities, IV_i, towards IV_0. When $a = 1$, we have $\tilde{IV}_i = IV_0\ \forall i$ (i.e., the transformed implied volatility curve is flat and takes on a value equal to the implied volatility of the OTM call option with the lowest implied volatility). When $a = 0$, we obtain the original implied volatilities (i.e., $\tilde{IV}_i = IV_i\ \forall i$).

The final ingredient in our heuristic correction is to select a. We do this in a conservative way that sets the risk-neutral market return skewness (approximately) to the physical skewness. Our baseline risk-neutral and physical skewness measures imply that risk-neutral skewness measured from observed options prices is approximately twice that of the physical skewness implied by our estimated preference parameters in Table 1. This holds unconditionally and is approximately true conditionally. As outside validation of this result, results reported in Beason and Schreindorfer (2020) imply that the ratio of unconditional risk-neutral to physical skewness is about 2. As an identifying assumption, we assume that this relationship holds conditionally and identify the a value such that this is true. We find that this holds on average when $a \approx 0.45$ and set a to this value in our analysis.

This implied volatility transformation attempts to match risk-neutral implied volatility and skewness to those from the physical market return distribution by shrinking the implied volatility curve towards IV_0. The idea is that any differences between IV_i and $\tilde{IV}_i$ represent options price premia that are the result of supply/demand imbalances unrelated to risk. To the extent that these price pressures exist, our transformation should produce implied options prices that are closer to their no-arbitrage benchmarks.

Given our estimated value of a, we re-estimate all risk-neutral moments using the transformed implied volatility curves, re-estimate preference parameters, and report the resulting unrestricted market risk premium decomposition in Figure IA.6 and Table IA.3. These results can be compared with our analogous main results in Figure 2 and Table 2. The modified decomposition produces results that are both qualitatively and quantitatively similar to our main results. The most noticeable difference is that this modification results in the central risk premium becoming a relatively larger component of the total risk premium at the expense of the downside risk premium. This is expected given that we have effectively decreased the implied risk-neutral skewness, making it more similar to the physical distribution skewness. The effect of this is to reduce the importance of the downside risk premium. It should be noted, though, that the downside risk premium still maintains a large contribution to the total risk premium. The upside risk premium contribution is very similar to in our main specification. The implied volatility transformation we impose in this subsection represents a drastic shift of the risk-neutral distribution towards the physical distribution (it effectively cuts the implied volatility smirk slope in half), yet the analysis produces similar results for the risk premium decomposition. We conclude that potential demand-driven options mispricing (or any other mispricing that generates an excessive implied volatility smirk) does not significantly alter our main results and conclusions.

IA.3 Utility Function-Implied Decompositions: Log, CRRA, CARA, and HARA Utilities

This section derives exact closed-form expressions for physical truncated moments in terms of risk neutral quantities when investor utility takes on various commonly specified functional forms. We focus on the log, CRRA, CARA, and HARA utility since these represent the most common forms of time-separable utility functions in extant literature. Given the utility-implied physical moments and corresponding risk-neutral moments, we can estimate risk premium decompositions implied under these various preference assumptions. Analytic expressions for the physical and risk-neutral moments as functions of option prices are provided in Internet Appendix IA.4. Note that, given any utility function, one can compute such closed-form expressions in terms of risk-neutral moments that can be estimated from option prices.

These formulations of our decomposition put slightly more structure on the decomposition than in the data-implied decomposition since they make assumptions about the functional form of representative investor preferences. Hence, they impose restrictions on the preference parameters that we estimate from the data for our unrestricted decomposition. They do not, however, impose as much structure as the representative agent models discussed in Section 3, which make assumptions about both the functional forms of preferences and state variable processes that govern the economy. Importantly, to derive results under these specific utility assumptions, we do not make any assumptions about the distribution of market returns or other state variables that govern the economy.

Remark 1. *Assume there exists a representative agent with log utility whose wealth is entirely invested in the market. Given no-arbitrage, the inverse SDF is given by*

$$\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}} = \frac{R_{M,t\to T}}{\mathbb{E}_t^*\left[R_{M,t\to T}\right]}$$

and the conditional physical truncated moments are given by

$$\mathbb{M}_{t\to T}^{(n)}\left[A_s\right] = \frac{1}{R_{f,t\to T}}\mathbb{M}_{t\to T}^{*(n+1)}\left[A_s\right] + \mathbb{M}_{t\to T}^{*(n)}\left[A_s\right] \tag{IA.1}$$

for any n and any $A_s \in \{A, A_d, A_{c,} A_u\}$. Closed form expressions for these risk-neutral moments are provided in Internet Appendix IA.4.

Proof. See Internet Appendix IA.4. ■

Remark 2. *Assume there exists a representative agent with CRRA utility over final wealth given by*

$$U\left(W_T\right) \quad = \quad \frac{W_T^{1-\alpha} - 1}{1-\alpha}$$

where α is the relative risk aversion, $W_T \equiv W_t R_{M,t\to T}$ is the final wealth, W_t is the initial wealth, and $R_{M,t\to T}$ is the return on the market. Assuming no-arbitrage, the inverse SDF is given by

$$\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}} = \frac{R_{M,t\to T}^{\alpha}}{\mathbb{E}_t^*\left[R_{M,t\to T}^{\alpha}\right]}$$

and the conditional physical truncated moments are given by

$$\mathbb{M}_{t\to T}^{(n)}\left[A_s\right] = \frac{\mathbb{E}_t^*\left[R_{M,t\to T}^{\alpha}\left(R_{M,t\to T} - R_{f,t\to T}\right)^n \mathbb{I}_{A_s}\right]}{\mathbb{E}_t^*\left[R_{M,t\to T}^{\alpha}\right]} \tag{IA.2}$$

for any n and any $A_s \in \{A, A_d, A_c, A_u\}$. Closed-form expressions for Equation IA.2 are provided in Internet Appendix IA.4.

Proof. See Internet Appendix IA.4. ∎

Remark 3. *Assume there exists a representative agent with CARA utility of the form*

$$U(W_T) = 1 - e^{-\tilde{\alpha} W_T}$$

where $\tilde{\alpha}$ is the absolute risk aversion, $W_T \equiv W_t R_{M,t\to T}$ is the final wealth, W_t is the initial wealth, and $R_{M,t\to T}$ is the return on the market. Define relative risk aversion as $\alpha \equiv \tilde{\alpha} W_t$. Assuming no-arbitrage, the inverse SDF is given by

$$\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}} = \frac{e^{\alpha R_{M,t\to T}}}{\mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\right]}$$

and the conditional physical truncated moments are given by

$$\mathbb{M}_{t\to T}^{(n)}\left[A_s\right] = \frac{\mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T} - R_{f,t\to T}\right)^n \mathbb{I}_{A_s}\right]}{\mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\right]} \tag{IA.3}$$

for any n any $A_s \in \{A, A_d, A_c, A_u\}$. Closed-form expressions for Equation IA.3 are provided in Internet Appendix IA.4.

Proof. See Internet Appendix IA.4. ∎

Remark 4. *Assume there exists a representative agent with HARA utility of the form*

$$U(W_T) = \frac{1-\gamma}{\gamma}\left(\frac{aW_T}{1-\gamma} + b\right)^{\gamma} \text{ with } a > 0 \text{ and } \frac{aW_T}{1-\gamma} + b > 0,$$

where $W_T \equiv W_t R_{M,t\to T}$ is the final wealth, W_t is the initial wealth, and $R_{M,t\to T}$ is the return on the market. Assuming no-arbitrage and decreasing relative risk aversion,[39] the inverse SDF is given by

$$\frac{\mathbb{E}_t\left[m_{t\to T}\right]}{m_{t\to T}} = \frac{\left(-a^*\left(R_{M,t\to T}/R_{f,t\to T}\right) - 1\right)^{1-\gamma}}{\mathbb{E}_t^*\left[\left(-a^*\left(R_{M,t\to T}/R_{f,t\to T}\right) - 1\right)^{1-\gamma}\right]}$$

where

$$a^* = \left(\frac{1-\gamma}{\mathcal{R}} - 1\right)^{-1}$$

[39] Decreasing relative risk aversion implies $b < 0$. We could also derive a similar expression assuming increasing relative risk aversion ($b > 0$), but choose to omit this less economically relevant case.

and $\mathcal{R}$ is the relative risk aversion evaluated at $W_t R_{f,t\to T}$. The conditional physical truncated moment is given by

$$\mathbb{M}^{(n)}_{t\to T}\left[A_s\right] = \frac{\mathbb{E}^*_t\left[\left(-a^*\left(R_{M,t\to T}/R_{f,t\to T}\right)-1\right)^{1-\gamma}\left(R_{M,t\to T}-R_{f,t\to T}\right)^n \mathbb{I}_{A_s}\right]}{\mathbb{E}^*_t\left[\left(-a^*\left(R_{M,t\to T}/R_{f,t\to T}\right)-1\right)^{1-\gamma}\right]} \tag{IA.4}$$

for any n any $A_s \in \{A, A_d, A_c, A_u\}$. The closed-form expression for Equation IA.4 is provided in Internet Appendix IA.4.

Proof. See Internet Appendix IA.4. ■

Given these physical moments and measured risk-neutral moments, we can compute our risk premium decomposition according to Equation 29 under these various preference specifications. Closed-form expressions that allow us to compute these moments directly from option prices are provided in Internet Appendix IA.4. Table IA.4 provides a summary of the average risk premium levels and contributions from each region under each set of preference assumptions. In the case of CRRA and CARA utility, we provide results for levels of relative risk aversion at three, five, and seven. In the case of HARA utility, we are limited to relatively low levels of risk aversion due to the functional form, and set these to 1.0, 1.1, and 1.2. Although the levels of risk premia implied by each specific utility function (Panel A) can be quite different than those from our data-implied decomposition, the contributions (Panel B) are actually more similar to those from the data-implied decomposition relative to those from the representative agent model-implied decompositions. This implies that the additional structure implied by these models can lead to model misspecification that has counterfactual implications for the relative contributions to the total risk premium to which the utility-based decomposition are immune.

IA.4 Expressions for Computing Risk-Neutral Moments

In this section, we use the Carr and Madan (2001) spanning formula to derive expressions for various risk-neutral moments needed for our decomposition as functions of observable options prices. The spanning formula can be written as:

$$h\left(y\right) \quad = \quad h\left(y_0\right) + \left(y-y_0\right) h_y\left(y_0\right) + \int_0^{y_0} h_{yy}\left(K\right)\left(K-y\right)^+ dK + \int_{y_0}^{\infty} h_{yy}\left(K\right)\left(y-K\right)^+ dK \tag{IA.5}$$

where $h\left(y\right)$ represents a generic function of y. We refer to y_0 as the "Carr and Madan expansion point." In our case, we are interested in functions of the future market index at time T, S_T (i.e., we set $y = S_T$). We can think of y_0 as a baseline market index value (e.g., $y_0 = R_{f,t\to T} S_t$, where S_t is the current market price at time t). We switch between using $R_{M,t\to T}$ with the equivalent expression S_T/S_t in this section when appropriate for clarity.

We also make use of the following indicator functions when computing truncated moments: $\mathbb{I}_{A_c} \equiv \mathbb{I}_{\{\underline{x} S_t \leq S_T \leq \overline{x} S_T\}}$, $\mathbb{I}_{A_d} \equiv \mathbb{I}_{\{S_T < \underline{x} S_t\}}$, and $\mathbb{I}_{A_u} \equiv \mathbb{I}_{\{S_T > \overline{x} S_t\}}$.

Expressions in this section require us to compute integrals of functions of option prices with respect to the strike price. Using the Black-Scholes formula, there is a one-to-one mapping between observed prices and implied volatilities. To compute risk-neutral moments we map observed prices to implied volatilities, fit the implied volatilities according to the procedure described in Footnote 13, then invert the fitted volatilities to obtain prices needed for the expressions in this section.

IA.4.1 Risk-Neutral Moments Centered on $R_{f,t\to T}$

In this subsection, we derive expressions for risk-neutral moments of the form $\mathbb{E}^* \left[\left(R_{M,t\to T} - R_{f,t\to T} \right)^n \right]$ and $\mathbb{E}^* \left[\left(R_{M,t\to T} - R_{f,t\to T} \right)^n \mathbb{I}_{A_s} \right]$ where $R_{M,t\to T}$ is the gross market return from time t to T, $R_{f,t\to T}$ is the risk-free rate from time t to T, and $\mathbb{I}_{A_s}$ is an indicator function for realized market returns belonging to sets A_s defined in Equations 4, 5, and 6. Note that we can express the gross market return as $R_{M,t\to T} = S_T/S_t$.

IA.4.1.1 Untruncated Risk-Neutral Moments Centered on $R_{f,t\to T}$: $\mathbb{E}^* \left[\left(R_{M,t\to T} - R_{f,t\to T} \right)^n \right]$

Set the function $h(\cdot)$ in Equation IA.5 to

$$h(S_T) = \left(\frac{S_T}{S_t} - R_{f,t\to T} \right)^n . \tag{IA.6}$$

Derivatives of this function are

$$h_y(S_T) = \frac{n}{S_t} \left(\frac{S_T}{S_t} - R_{f,t\to T} \right)^{n-1} \text{ and} \tag{IA.7}$$

$$h_{yy}(S_T) = \frac{n(n-1)}{S_t^2} \left(\frac{S_T}{S_t} - R_{f,t\to T} \right)^{n-2} . \tag{IA.8}$$

Next, set $y_0 = R_{f,t\to T} S_t$. Evaluating the function and its derivatives at values needed for Equation IA.5 yields:

$$h(R_{f,t\to T} S_t) = 0,$$

$$h_y(R_{f,t\to T} S_t) = 0, \text{ and}$$

$$h_{yy}(K) = \frac{n(n-1)}{S_t^2} \left(\frac{K}{S_t} - R_{f,t\to T} \right)^{n-2} .$$

Substituting these expressions into Equation IA.5 yields:

$$\begin{aligned}\left(\frac{S_T}{S_t}-R_{f,t\to T}\right)^n &= \frac{n\,(n-1)}{S_t^2}\int_0^{R_{f,t\to T}S_t}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}(K-S_T)^+\,dK\\ &\quad+\frac{n\,(n-1)}{S_t^2}\int_{R_{f,t\to T}S_t}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}(S_T-K)^+\,dK.\end{aligned}$$

Taking expectations under the risk-neutral measure at time t yields:

$$\begin{aligned}&\mathbb{E}_t^*\left[(R_{M,t\to T}-R_{f,t\to T})^n\right]\\ =\; &\frac{n\,(n-1)\,R_{f,t\to T}}{S_t^2}\int_0^{R_{f,t\to T}S_t}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}P_{t\to T}\left[K\right]dK\\ &+\frac{n\,(n-1)\,R_{f,t\to T}}{S_t^2}\int_{R_{f,t\to T}S_t}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}C_{t\to T}\left[K\right]dK\end{aligned}\tag{IA.9}$$

where $P_{t\to T}[K]$ and $C_{t\to T}[K]$ are the put and call prices with strike prices K at time t and expiration date T. Note that when $n=1$ the expression yields $\mathbb{E}_t^*\left[(R_{M,t\to T}-R_{f,t\to T})\right]=0$ as expected (i.e., because the risk-neutral expected market return is the risk-free rate).

IA.4.1.2 Downside Risk-Neutral Moments Centered on $R_{f,t\to T}$: $\mathbb{E}^*\left[(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_d}\right]$

Set the function $h(\cdot)$ in Equation IA.5 to that in Equation IA.6. Next, set $y_0=\underline{x}S_t$. Evaluating the function and its first derivative at values needed for Equation IA.5 yields:

$$\begin{aligned}h\left(\underline{x}S_t\right) &= \left(\underline{x}-R_{f,t\to T}\right)^n \text{ and}\\ h_y\left(\underline{x}S_t\right) &= \frac{n}{S_t}\left(\underline{x}-R_{f,t\to T}\right)^{n-1}.\end{aligned}$$

Substituting these expressions into Equation IA.5 yields:

$$\begin{aligned}\left(\frac{S_T}{S_t}-R_{f,t\to T}\right)^n &= \left(\underline{x}-R_{f,t\to T}\right)^n+n\left(\underline{x}-R_{f,t\to T}\right)^{n-1}\left(\frac{S_T}{S_t}-\underline{x}\right)\\ &\quad+\frac{n\,(n-1)}{S_t^2}\int_0^{\underline{x}S_t}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}(K-S_T)^+\,dK\\ &\quad+\frac{n\,(n-1)}{S_t^2}\int_{\underline{x}S_t}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}(S_T-K)^+\,dK.\end{aligned}$$

Multiplying by $\mathbb{I}_{\{S_T<\underline{x}S_t\}}$ yields:

$$
\begin{aligned}
\left(\frac{S_T}{S_t}-R_{f,t\to T}\right)^n \mathbb{I}_{\{S_T<\underline{x}S_t\}} \quad = \quad & \left(\underline{x}-R_{f,t\to T}\right)^n \mathbb{I}_{\{S_T<\underline{x}S_t\}} + n\left(\underline{x}-R_{f,t\to T}\right)^{n-1}\left(\frac{S_T}{S_t}-\underline{x}\right)\mathbb{I}_{\{S_T<\underline{x}S_t\}} \\
& +\frac{n\left(n-1\right)}{S_t^2}\int_0^{\underline{x}S_t}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(K-S_T\right)^+\mathbb{I}_{\{S_T<\underline{x}S_t\}}dK \\
& +\frac{n\left(n-1\right)}{S_t^2}\int_{\underline{x}S_t}^{\infty}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(S_T-K\right)^+\mathbb{I}_{\{S_T<\underline{x}S_t\}}dK.
\end{aligned}
$$

Simplifying this expression yields:

$$
\begin{aligned}
\left(\frac{S_T}{S_t}-R_{f,t\to T}\right)^n \mathbb{I}_{\{S_T<\underline{x}S_t\}} \quad = \quad & \left(\underline{x}-R_{f,t\to T}\right)^n \mathbb{I}_{\{S_T<\underline{x}S_t\}} - n\left(\underline{x}-R_{f,t\to T}\right)^{n-1}\frac{1}{S_t}\left(\underline{x}S_t-S_T\right)\mathbb{I}_{\{S_T<\underline{x}S_t\}} \\
& +\frac{n\left(n-1\right)}{S_t^2}\int_0^{\underline{x}S_t}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}\left(K-S_T\right)^+dK.
\end{aligned}
$$

Taking expectations under the risk-neutral measure at time t yields:

$$
\begin{aligned}
& \mathbb{E}^*\left[\left(R_{M,t\to T}-R_{f,t\to T}\right)^n \mathbb{I}_{A_d}\right] \\
= \quad & \left(\underline{x}-R_{f,t\to T}\right)^n Prob_t^*\left[R_{M,t\to T}<\underline{x}\right] - n\left(\underline{x}-R_{f,t\to T}\right)^{n-1}\frac{R_{f,t\to T}}{S_t}P_{t\to T}\left[\underline{x}S_t\right] \\
& +\frac{n\left(n-1\right)R_{f,t\to T}}{S_t^2}\int_0^{\underline{x}S_t}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}P_{t\to T}\left[K\right]dK
\end{aligned}
\tag{IA.10}
$$

where $Prob_t^*\left[R_{M,t\to T}<\underline{x}\right]$ is the risk-neutral probability at time t that $R_{M,t\to T}<\underline{x}$ and can be computed as:

$$
Prob_t^*\left[R_{M,t\to T}<\underline{x}\right] \quad = \quad R_{f,t\to T}\left.\frac{\partial P_{t\to T}\left[K\right]}{\partial K}\right|_{K=\underline{x}S_t}
\tag{IA.11}
$$

where $\left.\frac{\partial P_{t\to T}[K]}{\partial K}\right|_{K=\underline{x}S_t}$ is the partial derivative of the put price with respect to K evaluated at $K=\underline{x}S_t$. We also make use of the definitions $\mathbb{I}_{A_d}\equiv\mathbb{I}_{\{S_T<\underline{x}S_t\}}$ and $R_{M,t\to T}\equiv S_T/S_t$. We compute $\left.\frac{\partial P_{t\to T}[K]}{\partial K}\right|_{K=\underline{x}S_t}$ by computing the slope between put prices with strikes that span $\underline{x}S_t$.

IA.4.1.3 Upside Risk-Neutral Moments Centered on $R_{f,t\to T}$: $\mathbb{E}^*\left[\left(R_{M,t\to T}-R_{f,t\to T}\right)^n \mathbb{I}_{A_u}\right]$

Set the function $h\left(\cdot\right)$ in Equation IA.5 to that in Equation IA.6. Next, set $y_0=\overline{x}S_t$. Evaluating the function and its first derivative at values needed for Equation IA.5 yields:

$$
h\left(\overline{x}S_t\right) \quad = \quad \left(\overline{x}-R_{f,t\to T}\right)^n \text{ and}
$$

$$
h_y\left(\overline{x}S_t\right) \quad = \quad \frac{n}{S_t}\left(\overline{x}-R_{f,t\to T}\right)^{n-1}.
$$

Substituting these expressions into Equation IA.5 yields:

$$\begin{aligned}\left(\frac{S_T}{S_t} - R_{f,t\to T}\right)^n &= \left(\overline{x} - R_{f,t\to T}\right)^n + n\left(\overline{x} - R_{f,t\to T}\right)^{n-1}\left(\frac{S_T}{S_t} - \overline{x}\right) \\ &+ \frac{n(n-1)}{S_t^2}\int_0^{\overline{x}S_t}\left(\frac{K}{S_t} - R_{f,t\to T}\right)^{n-2}(K - S_T)^+\,dK \\ &+ \frac{n(n-1)}{S_t^2}\int_{\overline{x}S_t}^{\infty}\left(\frac{K}{S_t} - R_{f,t\to T}\right)^{n-2}(S_T - K)^+\,dK.\end{aligned}$$

Multiplying by $\mathbb{I}_{\{S_T > \overline{x}S_t\}}$ yields:

$$\begin{aligned}\left(\frac{S_T}{S_t} - R_{f,t\to T}\right)^n \mathbb{I}_{\{S_T > \overline{x}S_t\}} &= \left(\overline{x} - R_{f,t\to T}\right)^n \mathbb{I}_{\{S_T > \overline{x}S_t\}} + n\left(\overline{x} - R_{f,t\to T}\right)^{n-1}\left(\frac{S_T}{S_t} - \overline{x}\right)\mathbb{I}_{\{S_T > \overline{x}S_t\}} \\ &+ \frac{n(n-1)}{S_t^2}\int_0^{\overline{x}S_t}\left(\frac{K}{S_t} - R_{f,t\to T}\right)^{n-2}(K - S_T)^+\,\mathbb{I}_{\{S_T > \overline{x}S_t\}}dK \\ &+ \frac{n(n-1)}{S_t^2}\int_{\overline{x}S_t}^{\infty}\left(\frac{K}{S_t} - R_{f,t\to T}\right)^{n-2}(S_T - K)^+\,\mathbb{I}_{\{S_T > \overline{x}S_t\}}dK.\end{aligned}$$

Simplifying this expression yields:

$$\begin{aligned}\left(\frac{S_T}{S_t} - R_{f,t\to T}\right)^n \mathbb{I}_{\{S_T > \overline{x}S_t\}} &= \left(\overline{x} - R_{f,t\to T}\right)^n \mathbb{I}_{\{S_T > \overline{x}S_t\}} + n\left(\overline{x} - R_{f,t\to T}\right)^{n-1}\frac{1}{S_t}\left(S_T - \overline{x}S_t\right)^+ \\ &+ \frac{n(n-1)}{S_t^2}\int_{\overline{x}S_t}^{\infty}\left(\frac{K}{S_t} - R_{f,t\to T}\right)^{n-2}(S_T - K)^+\,dK.\end{aligned}$$

Taking expectations under the risk-neutral measure at time t yields:

$$\begin{aligned}&\mathbb{E}^*\left[\left(R_{M,t\to T} - R_{f,t\to T}\right)^n \mathbb{I}_{A_u}\right] \\ =\ &\left(\overline{x} - R_{f,t\to T}\right)^n Prob_t^*\left[R_{M,t\to T} > \overline{x}\right] + n\left(\overline{x} - R_{f,t\to T}\right)^{n-1}\frac{R_{f,t\to T}}{S_t}C_{t\to T}\left[\overline{x}S_t\right] \\ &+ \frac{n(n-1)R_{f,t\to T}}{S_t^2}\int_{\overline{x}S_t}^{\infty}\left(\frac{K}{S_t} - R_{f,t\to T}\right)^{n-2}C_{t\to T}[K]\,dK\end{aligned} \tag{IA.12}$$

where $Prob_t^*\left[R_{M,t\to T} > \overline{x}\right]$ is the risk-neutral probability at time t that $R_{M,t\to T} > \overline{x}$ and can be computed as:

$$Prob_t^*\left[R_{M,t\to T} > \overline{x}\right] = -R_{f,t\to T}\left.\frac{\partial C_{t\to T}[K]}{\partial K}\right|_{K=\overline{x}S_t} \tag{IA.13}$$

where $\left.\frac{\partial C_{t\to T}[K]}{\partial K}\right|_{K=\overline{x}S_t}$ is the partial derivative of the call price with respect to K evaluated at $K = \overline{x}S_t$. We also make use of the definitions $\mathbb{I}_{A_u} \equiv \mathbb{I}_{\{S_T > \overline{x}S_t\}}$ and $R_{M,t\to T} \equiv S_T/S_t$. We compute $\left.\frac{\partial C_{t\to T}[K]}{\partial K}\right|_{K=\overline{x}S_t}$ by computing the slope between call prices with strikes that span $\overline{x}S_t$.

IA.4.1.4 Central Risk-Neutral Moments Centered on $R_{f,t\to T}$: $\mathbb{E}^*\left[(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_c}\right]$

Observe the following identity:

$$\mathbb{I}_{\{\underline{x}S_t\leq S_T\leq\overline{x}S_T\}} \equiv 1-\mathbb{I}_{\{S_T<\underline{x}S_t\}}-\mathbb{I}_{\{S_T>\overline{x}S_t\}}. \tag{IA.14}$$

This identity implies the following identity relating the risk-neutral moments:

$$\begin{aligned}\mathbb{E}^*\left[(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_c}\right] \equiv\ & \mathbb{E}_t^*\left[(R_{M,t\to T}-R_{f,t\to T})^n\right]\\ &-\mathbb{E}^*\left[(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_d}\right]\\ &-\mathbb{E}^*\left[(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_u}\right]\end{aligned}$$

where we have made use of the definitions $\mathbb{I}_{A_c}\equiv\mathbb{I}_{\{\underline{x}S_t\leq S_T\leq\overline{x}S_T\}}$, $\mathbb{I}_{A_d}\equiv\mathbb{I}_{\{S_T<\underline{x}S_t\}}$, and $\mathbb{I}_{A_u}\equiv\mathbb{I}_{\{S_T>\overline{x}S_t\}}$. Substituting in expressions from Equations IA.9, IA.10, and IA.12 and simplifying yields:

$$\begin{aligned}&\mathbb{E}^*\left[(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_c}\right]\\ =\ & -\left(\underline{x}-R_{f,t\to T}\right)^n Prob_t^*\left[S_T<\underline{x}S_t\right]-\left(\overline{x}-R_{f,t\to T}\right)^n Prob_t^*\left[S_T>\overline{x}S_t\right]\\ &+n\left(\underline{x}-R_{f,t\to T}\right)^{n-1}\frac{R_{f,t\to T}}{S_t}P_{t\to T}\left[\underline{x}S_t\right]-n\left(\overline{x}-R_{f,t\to T}\right)^{n-1}\frac{R_{f,t\to T}}{S_t}C_{t\to T}\left[\overline{x}S_t\right]\\ &+\frac{n\,(n-1)\,R_{f,t\to T}}{S_t^2}\left[\int_{\underline{x}S_t}^{R_{f,t\to T}S_t}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}P_{t\to T}\left[K\right]dK\right]\\ &+\frac{n\,(n-1)\,R_{f,t\to T}}{S_t^2}\left[\int_{R_{f,t\to T}S_t}^{\overline{x}S_t}\left(\frac{K}{S_t}-R_{f,t\to T}\right)^{n-2}C_{t\to T}\left[K\right]dK\right].\end{aligned} \tag{IA.15}$$

IA.4.2 Risk-Neutral Moments for log Utility-Based Physical Moments

Proof. Proof of Remark 1. Assuming no-arbitrage conditions, we can show

$$\begin{aligned}\mathbb{M}_{t\to T}^{(n)}\left[A_s\right] &\equiv \mathbb{E}_t\left[\frac{M_{t\to T}}{\mathbb{E}_t\left[M_{t\to T}\right]}\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}}\left(R_{M,t\to T}-R_{f,t\to T}\right)^n\mathbb{I}_{A_s}\right]\\ &= \mathbb{E}_t^*\left[\frac{\mathbb{E}_t\left[M_{t\to T}\right]}{M_{t\to T}}\left(R_{M,t\to T}-R_{f,t\to T}\right)^n\mathbb{I}_{A_s}\right]\\ &= \frac{\mathbb{E}_t^*\left[R_{M,t\to T}\left(R_{M,t\to T}-R_{f,t\to T}\right)^n\mathbb{I}_{A_s}\right]}{\mathbb{E}_t^*\left[R_{M,t\to T}\right]}\\ &= \frac{\mathbb{E}_t^*\left[\left(R_{M,t\to T}-R_{f,t\to T}+R_{f,t\to T}\right)\left(R_{M,t\to T}-R_{f,t\to T}\right)^n\mathbb{I}_{A_s}\right]}{\mathbb{E}_t^*\left[R_{M,t\to T}\right]}\\ &= \frac{\mathbb{M}_{t\to T}^{*(n+1)}\left[A_s\right]}{R_{f,t\to T}}+\mathbb{M}_{t\to T}^{*(n)}\left[A_s\right].\end{aligned} \tag{IA.16}$$

■

IA.4.3 Risk-Neutral Moments for CRRA Utility-Based Physical Moments

Proof. Proof of Remark 2. Assuming no-arbitrage conditions, we can show

$$\begin{aligned}
\mathbb{M}_{t\to T}^{(n)}[A_s] &\equiv \mathbb{E}_t\left[\frac{M_{t\to T}}{\mathbb{E}_t[M_{t\to T}]}\frac{\mathbb{E}_t[m_{t\to T}]}{m_{t\to T}}(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_s}\right] \\
&= \mathbb{E}_t^*\left[\frac{\mathbb{E}_t[m_{t\to T}]}{m_{t\to T}}(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_s}\right] \\
&= \frac{\mathbb{E}_t^*\left[R_{M,t\to T}^{\alpha}(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_s}\right]}{\mathbb{E}_t^*\left[R_{M,t\to T}^{\alpha}\right]}. \qquad \text{(IA.17)}
\end{aligned}$$

The second expression is obtained by replacing the inverse of the SDF by its expression. This ends the proof.

■

We then show how to compute risk-neutral moments in Remark 2 (Equation IA.2). Specifically, we would like to compute moments of the form $\mathbb{E}_t^*\left[(R_{M,t\to T})^{\alpha}\left(\frac{S_T}{S_t}-R_{f,t\to T}\right)^n\mathbb{I}_{A_s}\right]$. We first apply the binomial theorem to show:

$$\begin{aligned}
&= \mathbb{E}_t^*\left[(R_{M,t\to T})^{\alpha}(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_s}\right] \\
&\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!\,(n-k)!}(R_{f,t\to T})^{n-k}\,\mathbb{E}_t^*\left[(R_{M,t\to T})^{k+\alpha}\,\mathbb{I}_{A_s}\right]. \qquad \text{(IA.18)}
\end{aligned}$$

So we need only compute moments of $(R_{M,t\to T})^{k+\alpha}\,\mathbb{I}_{A_s}$ in order to construct moments of $(R_{M,t\to T})^{\alpha}(R_{M,t\to T}-R_{f,t\to T})^n\,\mathbb{I}_{A_s}$. We again make use of the Carr and Madan (2001) spanning formula (Equation IA.5) to compute these moments as functions of options prices.

IA.4.3.1 CRRA: Untruncated Risk-Neutral Moments: $\mathbb{E}_t^*\left[(R_{M,t\to T})^{k+\alpha}\right]$

Set the function $h(\cdot)$ in Equation IA.5 to

$$h^{CRRA}(S_T) = \left(\frac{S_T}{S_t}\right)^{k+\alpha}. \qquad \text{(IA.19)}$$

Derivatives of this function are

$$h_y^{CRRA}(S_T) = \frac{k+\alpha}{S_t}\left(\frac{S_T}{S_t}\right)^{k+\alpha-1} \quad \text{and} \qquad \text{(IA.20)}$$

$$h_{yy}^{CRRA}(S_T) = \frac{(k+\alpha)(k+\alpha-1)}{S_t^2}\left(\frac{S_T}{S_t}\right)^{k+\alpha-2}. \qquad \text{(IA.21)}$$

Next, set $y_0 = R_{f,t\to T}S_t$. Evaluating the function and its derivatives at values needed for Equation IA.5 yields:

$$h^{CRRA}\left(R_{f,t\to T}S_t\right) = \left(R_{f,t\to T}\right)^{k+\alpha},$$

$$h_y^{CRRA}\left(R_{f,t\to T}S_t\right) = \frac{k+\alpha}{S_t}\left(R_{f,t\to T}\right)^{k+\alpha-1}, \text{ and}$$

$$h_{yy}^{CRRA}\left(K\right) = \frac{\left(k+\alpha\right)\left(k+\alpha-1\right)}{S_t^2}\left(\frac{K}{S_t}\right)^{k+\alpha-2}.$$

Substituting these expressions into Equation IA.5 yields:

$$\begin{aligned}\left(\frac{S_T}{S_t}\right)^{k+\alpha} &= \left(R_{f,t\to T}\right)^{k+\alpha} + \left(k+\alpha\right)\left(\frac{S_T}{S_t} - R_{f,t\to T}\right)\left(R_{f,t\to T}\right)^{k+\alpha-1} \\ &+ \frac{\left(k+\alpha\right)\left(k+\alpha-1\right)}{S_t^2}\int_0^{R_{f,t\to T}S_t}\left(\frac{K}{S_t}\right)^{k+\alpha-2}\left(K-S_T\right)^+ dK \\ &+ \frac{\left(k+\alpha\right)\left(k+\alpha-1\right)}{S_t^2}\int_{R_{f,t\to T}S_t}^{\infty}\left(\frac{K}{S_t}\right)^{k+\alpha-2}\left(S_T-K\right)^+ dK.\end{aligned}$$

Taking expectations under the risk-neutral measure at time t yields:

$$\begin{aligned}&\mathbb{E}_t^*\left[\left(R_{M,t\to T}\right)^{k+\alpha}\right] \\ =\ &\left(R_{f,t\to T}\right)^{k+\alpha} \\ &+ \frac{\left(k+\alpha\right)\left(k+\alpha-1\right)R_{f,t\to T}}{S_t^2}\int_0^{R_{f,t\to T}S_t}\left(\frac{K}{S_t}\right)^{k+\alpha-2}P_{t\to T}\left[K\right]dK \\ &+ \frac{\left(k+\alpha\right)\left(k+\alpha-1\right)R_{f,t\to T}}{S_t^2}\int_{R_{f,t\to T}S_t}^{\infty}\left(\frac{K}{S_t}\right)^{k+\alpha-2}C_{t\to T}\left[K\right]dK.\end{aligned} \tag{IA.22}$$

Equation IA.22 can be combined with Equation IA.18 to compute the risk-neutral moments required for Equation IA.2 in Remark 2 when $\mathbb{I}_{A_s} = 1$ (i.e., the untruncated moment case).

IA.4.3.2 CRRA: Downside Risk-Neutral Moments: $\mathbb{E}_t^*\left[\left(R_{M,t\to T}\right)^{k+\alpha}\mathbb{I}_{A_d}\right]$

Set the function $h\left(\cdot\right)$ in Equation IA.5 to that in Equation IA.19. Next, set $y_0 = \underline{x}S_t$. Evaluating the function and its first derivative at values needed for Equation IA.5 yields:

$$h^{CRRA}\left(\underline{x}S_t\right) = \left(\underline{x}\right)^{k+\alpha} \text{ and}$$

$$h_y^{CRRA}\left(\underline{x}S_t\right) = \frac{k+\alpha}{S_t}\left(\underline{x}\right)^{k+\alpha-1}.$$

Note that $h_{yy}^{CRRA}(K)$ based on Equation IA.21 is unchanged. We can substitute these into the Carr and Madan (2001) spanning formula (Equation IA.5) and multiply the whole equation by $\mathbb{I}_{\{S_T<\underline{x}S_t\}}$ to obtain:

$$\begin{aligned}\left(\frac{S_T}{S_t}\right)^{k+\alpha}\mathbb{I}_{\{S_T<\underline{x}S_t\}} &= (\underline{x})^{k+\alpha}\mathbb{I}_{\{S_T<\underline{x}S_t\}}+(S_T-\underline{x}S_t)\frac{k+\alpha}{S_t}(\underline{x})^{k+\alpha-1}\mathbb{I}_{\{S_T<\underline{x}S_t\}}\\ &+\frac{(k+\alpha)(k+\alpha-1)}{S_t^2}\int_0^{\underline{x}S_t}\left(\frac{K}{S_t}\right)^{k+\alpha-2}(K-S_T)^+\mathbb{I}_{\{S_T<\underline{x}S_t\}}dK\\ &+\frac{(k+\alpha)(k+\alpha-1)}{S_t^2}\int_{\underline{x}S_t}^{\infty}\left(\frac{K}{S_t}\right)^{k+\alpha-2}(S_T-K)^+\mathbb{I}_{\{S_T<\underline{x}S_t\}}dK.\end{aligned}$$

Rearranging and simplifying (noting that the second integral is zero) yields:

$$\begin{aligned}\left(\frac{S_T}{S_t}\right)^{k+\alpha}\mathbb{I}_{\{S_T<\underline{x}S_t\}} &= (\underline{x})^{k+\alpha}\mathbb{I}_{\{S_T<\underline{x}S_t\}}-(k+\alpha)(\underline{x})^{k+\alpha-1}\frac{1}{S_t}(\underline{x}S_t-S_T)^+\\ &+\frac{(k+\alpha)(k+\alpha-1)}{S_t^2}\int_0^{\underline{x}S_t}\left(\frac{K}{S_t}\right)^{k+\alpha-2}(K-S_T)^+dK.\end{aligned}$$

Taking expectations under the risk-neutral measure yields:

$$\begin{aligned}&\mathbb{E}_t^*\left[(R_{M,t\to T})^{k+\alpha}\mathbb{I}_{A_d}\right]\\ =\ &(\underline{x})^{k+\alpha}Prob_t^*[S_T<\underline{x}S_t]-(k+\alpha)(\underline{x})^{k+\alpha-1}\frac{R_{f,t\to T}}{S_t}P_{t\to T}[\underline{x}S_t]\\ &+\frac{(k+\alpha)(k+\alpha-1)R_{f,t\to T}}{S_t^2}\int_0^{\underline{x}S_t}\left(\frac{K}{S_t}\right)^{k+\alpha-2}P_{t\to T}[K]\,dK.\end{aligned}\tag{IA.23}$$

Equation IA.23 can be combined with Equation IA.18 to compute the risk-neutral moments required for Equation IA.2 in Remark 2 with $\mathbb{I}_{A_s}=\mathbb{I}_{A_d}$.

IA.4.3.3 CRRA: Upside Risk-Neutral Moments: $\mathbb{E}_t^*\left[(R_{M,t\to T})^{k+\alpha}\mathbb{I}_{A_u}\right]$

Set the function $h(\cdot)$ in Equation IA.5 to that in Equation IA.19. Next, set $y_0=\overline{x}S_t$. Evaluating the function and its first derivative at values needed for Equation IA.5 yields:

$$h^{CRRA}(\overline{x}S_t) = (\overline{x})^{k+\alpha}\text{ and}$$

$$h_y^{CRRA}(\overline{x}S_t) = \frac{k+\alpha}{S_t}(\overline{x})^{k+\alpha-1}.$$

Note that $h_{yy}^{CRRA}(K)$ based on Equation IA.21 is unchanged. We can substitute these into the Carr and Madan (2001) spanning formula (Equation IA.5) and multiply the whole equation by $\mathbb{I}_{\{S_T > \overline{x} S_t\}}$ to obtain:

$$\begin{aligned}\left(\frac{S_T}{S_t}\right)^{k+\alpha} \mathbb{I}_{\{S_T > \overline{x} S_t\}} &= (\overline{x})^{k+\alpha} \mathbb{I}_{\{S_T > \overline{x} S_t\}} + (S_T - \overline{x} S_t) \frac{k+\alpha}{S_t} (\overline{x})^{k+\alpha-1} \mathbb{I}_{\{S_T > \overline{x} S_t\}} \\ &\quad + \frac{(k+\alpha)(k+\alpha-1)}{S_t^2} \int_0^{\overline{x} S_t} \left(\frac{K}{S_t}\right)^{k+\alpha-2} (K - S_T)^+ \mathbb{I}_{\{S_T > \overline{x} S_t\}} dK \\ &\quad + \frac{(k+\alpha)(k+\alpha-1)}{S_t^2} \int_{\overline{x} S_t}^{\infty} \left(\frac{K}{S_t}\right)^{k+\alpha-2} (S_T - K)^+ \mathbb{I}_{\{S_T > \overline{x} S_t\}} dK.\end{aligned}$$

Rearranging and simplifying (noting that the first integral is zero) yields:

$$\begin{aligned}\left(\frac{S_T}{S_t}\right)^{k+\alpha} \mathbb{I}_{\{S_T > \overline{x} S_t\}} &= (\overline{x})^{k+\alpha} \mathbb{I}_{\{S_T > \overline{x} S_t\}} + \frac{k+\alpha}{S_t} (\overline{x})^{k+\alpha-1} (S_T - \overline{x} S_t)^+ \\ &\quad + \frac{(k+\alpha)(k+\alpha-1)}{S_t^2} \int_{\overline{x} S_t}^{\infty} \left(\frac{K}{S_t}\right)^{k+\alpha-2} (S_T - K)^+ dK.\end{aligned}$$

Taking expectations under the risk-neutral measure yields:

$$\begin{aligned}&\mathbb{E}_t^*\left[(R_{M,t\to T})^{k+\alpha} \mathbb{I}_{A_u}\right] \\ = \ &(\overline{x})^{k+\alpha} Prob_t^*\left[S_T > \overline{x} S_t\right] + (k+\alpha)(\overline{x})^{k+\alpha-1} \frac{R_{f,t\to T}}{S_t} C_{t\to T}\left[\overline{x} S_t\right] \\ &+ \frac{(k+\alpha)(k+\alpha-1) R_{f,t\to T}}{S_t^2} \int_{\overline{x} S_t}^{\infty} \left(\frac{K}{S_t}\right)^{k+\alpha-2} C_{t\to T}[K] \, dK.\end{aligned} \tag{IA.24}$$

Equation IA.24 can be combined with Equation IA.18 to compute the risk-neutral moments required for Equation IA.2 in Remark 2 with $\mathbb{I}_{A_s} = \mathbb{I}_{A_u}$.

IA.4.3.4 CRRA: Central Risk-Neutral Moments: $\mathbb{E}_t^*\left[(R_{M,t\to T})^{k+\alpha} \mathbb{I}_{A_c}\right]$

The identity in Equation IA.14 implies the following identity relating the risk-neutral moments:

$$\begin{aligned}\mathbb{E}^*\left[(R_{M,t\to T})^{k+\alpha} \mathbb{I}_{A_c}\right] &\equiv \mathbb{E}_t^*\left[(R_{M,t\to T})^{k+\alpha}\right] \\ &\quad - \mathbb{E}^*\left[(R_{M,t\to T})^{k+\alpha} \mathbb{I}_{A_d}\right] \\ &\quad - \mathbb{E}^*\left[(R_{M,t\to T})^{k+\alpha} \mathbb{I}_{A_u}\right]\end{aligned}$$

where we have made use of the definitions $\mathbb{I}_{A_c} \equiv \mathbb{I}_{\{\underline{x} S_t \le S_T \le \overline{x} S_T\}}$, $\mathbb{I}_{A_d} \equiv \mathbb{I}_{\{S_T < \underline{x} S_t\}}$, and $\mathbb{I}_{A_u} \equiv \mathbb{I}_{\{S_T > \overline{x} S_t\}}$. Substituting in expressions from Equations IA.22, IA.23, and IA.24 and simplifying yields:

$$
\begin{aligned}
& \mathbb{E}^*\left[\left(R_{M,t\to T}\right)^{k+\alpha}\mathbb{I}_{A_c}\right] \\
= \; & \left(R_{f,t\to T}\right)^{k+\alpha}-\left(\underline{x}\right)^{k+\alpha} Prob_t^*\left[S_T<\underline{x}S_t\right]-\left(\overline{x}\right)^{k+\alpha} Prob_t^*\left[S_T>\overline{x}S_t\right] \\
& +\left(k+\alpha\right)\left(\underline{x}\right)^{k+\alpha-1}\frac{R_{f,t\to T}}{S_t}P_{t\to T}\left[\underline{x}S_t\right]-\left(k+\alpha\right)\left(\overline{x}\right)^{k+\alpha-1}\frac{R_{f,t\to T}}{S_t}C_{t\to T}\left[\overline{x}S_t\right] \\
& +\frac{\left(k+\alpha\right)\left(k+\alpha-1\right)R_{f,t\to T}}{S_t^2}\int_{\underline{x}S_t}^{R_{f,t\to T}S_t}\left(\frac{K}{S_t}\right)^{k+\alpha-2}P_{t\to T}\left[K\right]dK \\
& +\frac{\left(k+\alpha\right)\left(k+\alpha-1\right)R_{f,t\to T}}{S_t^2}\int_{R_{f,t\to T}S_t}^{\overline{x}S_t}\left(\frac{K}{S_t}\right)^{k+\alpha-2}C_{t\to T}\left[K\right]dK. \qquad \text{(IA.25)}
\end{aligned}
$$

Equation IA.25 can be combined with Equation IA.18 to compute the risk-neutral moments required for Equation IA.2 in Remark 2 with $\mathbb{I}_{A_s}=\mathbb{I}_{A_c}$.

IA.4.4 Risk-Neutral Moments for CARA Utility-Based Physical Moments

Proof. Proof of Remark 3. The conditional truncated moment is

$$
\begin{aligned}
\mathbb{M}_{t\to T}^{(n)}\left[A_s\right] &= \mathbb{E}_t^*\left[\frac{\mathbb{E}_t\left[m_{t\to T}\right]}{m_{t\to T}}\left(R_{M,t\to T}-R_{f,t\to T}\right)^n\mathbb{I}_{A_s}\right] \\
&= \mathbb{E}_t^*\left[\frac{e^{\alpha R_{M,t\to T}}}{\mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\right]}\left(R_{M,t\to T}-R_{f,t\to T}\right)^n\mathbb{I}_{A_s}\right] \\
&= \frac{\mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}-R_{f,t\to T}\right)^n\mathbb{I}_{A_s}\right]}{\mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\right]}. \qquad \text{(IA.26)}
\end{aligned}
$$

This ends the proof. ■

The last expression is obtained by replacing the inverse of the SDF by its expression. In this sub-section, we show how to compute risk-neutral moments in Remark 3 (Equation IA.3). Specifically, we would like to compute moments of the form $\mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}-R_{f,t\to T}\right)^n\mathbb{I}_{A_s}\right]$. We first apply the binomial theorem to show:

$$
\begin{aligned}
& \mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}-R_{f,t\to T}\right)^n\mathbb{I}_{A_s}\right] \\
= \; & \sum_{k=0}^{n}\left(-1\right)^{n-k}\frac{n!}{k!\left(n-k\right)!}\left(R_{f,t\to T}\right)^{n-k}\mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}\right)^k\mathbb{I}_{A_s}\right]. \qquad \text{(IA.27)}
\end{aligned}
$$

So we need only compute moments of $e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}\right)^k\mathbb{I}_{A_s}$ in order to construct moments of $e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}-R_{f,t\to T}\right)^n\mathbb{I}_{A_s}$. We again make use of the Carr and Madan (2001) spanning formula (Equation IA.5) to compute these moments as functions of options prices.

IA.4.4.1 CARA: Untruncated Risk-Neutral Moments: $\mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}\right)^k\right]$

Set the function $h(\cdot)$ in Equation IA.5 to

$$h^{CARA}(S_T) = e^{\alpha\frac{S_T}{S_t}}\left(\frac{S_T}{S_t}\right)^k. \tag{IA.28}$$

Derivatives of this function are

$$h_y^{CARA}(S_T) = \frac{\alpha}{S_t}e^{\alpha\frac{S_T}{S_t}}\left(\frac{S_T}{S_t}\right)^k + \frac{k}{S_t}e^{\alpha\frac{S_T}{S_t}}\left(\frac{S_T}{S_t}\right)^{k-1} \text{ and} \tag{IA.29}$$

$$\begin{aligned} h_{yy}^{CARA}(S_T) &= \frac{\alpha^2}{S_t^2}e^{\alpha\frac{S_T}{S_t}}\left(\frac{S_T}{S_t}\right)^k + 2\frac{\alpha k}{S_t^2}e^{\alpha\frac{S_T}{S_t}}\left(\frac{S_T}{S_t}\right)^{k-1} \\ &\quad + \frac{k(k-1)}{S_t^2}e^{\alpha\frac{S_T}{S_t}}\left(\frac{S_T}{S_t}\right)^{k-2}. \end{aligned} \tag{IA.30}$$

Next, set $y_0 = R_{f,t\to T}S_t$. Evaluating the function and its derivatives at values needed for Equation IA.5 yields:

$$h^{CARA}(R_{f,t\to T}S_t) = e^{\alpha R_{f,t\to T}}\left(R_{f,t\to T}\right)^k,$$

$$h_y^{CARA}(R_{f,t\to T}S_t) = \frac{\alpha}{S_t}e^{\alpha R_{f,t\to T}}\left(R_{f,t\to T}\right)^k + \frac{k}{S_t}e^{\alpha R_{f,t\to T}}\left(R_{f,t\to T}\right)^{k-1}, \text{ and}$$

$$\begin{aligned} h_{yy}^{CARA}(K) &= \frac{\alpha^2}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^k + 2\frac{\alpha k}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-1} \\ &\quad + \frac{k(k-1)}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-2}. \end{aligned}$$

Substituting these expressions into Equation IA.5 yields:

$$\begin{aligned} & e^{\alpha\frac{S_T}{S_t}}\left(\frac{S_T}{S_t}\right)^k \\ = \;& e^{\alpha R_{f,t\to T}}\left(R_{f,t\to T}\right)^k + \left(S_T - R_{f,t\to T}S_t\right)\left[\frac{\alpha}{S_t}e^{\alpha R_{f,t\to T}}\left(R_{f,t\to T}\right)^k + \frac{k}{S_t}e^{\alpha R_{f,t\to T}}\left(R_{f,t\to T}\right)^{k-1}\right] \\ & + \int_0^{R_{f,t\to T}S_t}\left[\frac{\alpha^2}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^k + 2\frac{\alpha k}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-1} + \frac{k(k-1)}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-2}\right]\left(K - S_T\right)^+ dK \\ & + \int_{R_{f,t\to T}S_t}^{\infty}\left[\frac{\alpha^2}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^k + 2\frac{\alpha k}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-1} + \frac{k(k-1)}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-2}\right]\left(S_T - K\right)^+ dK. \end{aligned}$$

Taking expectations under the risk-neutral measure at time t yields:

$$\begin{aligned}
& \mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}\right)^k\right] \\
= \quad & e^{\alpha R_{f,t\to T}}\left(R_{f,t\to T}\right)^k \\
& +\frac{R_{f,t\to T}}{S_t^2}\int_0^{R_{f,t\to T}S_t} e^{\alpha\frac{K}{S_t}}\left[\alpha^2\left(\frac{K}{S_t}\right)^k+2\alpha k\left(\frac{K}{S_t}\right)^{k-1}+k\left(k-1\right)\left(\frac{K}{S_t}\right)^{k-2}\right]P_{t\to T}\left[K\right]dK \\
& +\frac{R_{f,t\to T}}{S_t^2}\int_{R_{f,t\to T}S_t}^{\infty} e^{\alpha\frac{K}{S_t}}\left[\alpha^2\left(\frac{K}{S_t}\right)^k+2\alpha k\left(\frac{K}{S_t}\right)^{k-1}+k\left(k-1\right)\left(\frac{K}{S_t}\right)^{k-2}\right]C_{t\to T}\left[K\right]dK
\end{aligned} \tag{IA.31}$$

Equation IA.31 can be combined with Equation IA.27 to compute the risk-neutral moments required for Equation IA.3 in Remark 3 when $\mathbb{I}_{A_s}=1$ (i.e., the untruncated moment case).

IA.4.4.2 CARA: Downside Risk-Neutral Moments: $\mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}\right)^k\mathbb{I}_{A_d}\right]$

Set the function $h\left(\cdot\right)$ in Equation IA.5 to that in Equation IA.28. Next, set $y_0=\underline{x}S_t$. Evaluating the function and its first derivative at values needed for Equation IA.5 yields:

$$h^{CARA}\left(\underline{x}S_t\right) = e^{\alpha\underline{x}}\left(\underline{x}\right)^k \text{ and}$$

$$h_y^{CARA}\left(\underline{x}S_t\right) = \frac{\alpha}{S_t}e^{\alpha\underline{x}}\left(\underline{x}\right)^k+\frac{k}{S_t}e^{\alpha\underline{x}}\left(\underline{x}\right)^{k-1}.$$

Note that $h_{yy}^{CARA}\left(K\right)$ based on Equation IA.30 is unchanged. We can substitute these into the Carr and Madan (2001) spanning formula (Equation IA.5) and multiply the whole equation by $\mathbb{I}_{\{S_T<\underline{x}S_t\}}$ to obtain:

$$\begin{aligned}
& e^{\alpha\frac{S_T}{S_t}}\left(\frac{S_T}{S_t}\right)^k\mathbb{I}_{\{S_T<\underline{x}S_t\}} \\
= \quad & e^{\alpha\underline{x}}\left(\underline{x}\right)^k\mathbb{I}_{\{S_T<\underline{x}S_t\}}+\left(S_T-\underline{x}S_t\right)\left[\frac{\alpha}{S_t}e^{\alpha\underline{x}}\left(\underline{x}\right)^k+\frac{k}{S_t}e^{\alpha\underline{x}}\left(\underline{x}\right)^{k-1}\right]\mathbb{I}_{\{S_T<\underline{x}S_t\}} \\
& +\int_0^{\underline{x}S_t}\left[\frac{\alpha^2}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^k+2\frac{\alpha k}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-1}+\frac{k\left(k-1\right)}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-2}\right]\left(K-S_T\right)^+\mathbb{I}_{\{S_T<\underline{x}S_t\}}dK \\
& +\int_{\underline{x}S_t}^{\infty}\left[\frac{\alpha^2}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^k+2\frac{\alpha k}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-1}+\frac{k\left(k-1\right)}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-2}\right]\left(S_T-K\right)^+\mathbb{I}_{\{S_T<\underline{x}S_t\}}dK.
\end{aligned}$$

Rearranging and simplifying (noting that the second integral is zero) yields:

$$\begin{aligned}
& e^{\alpha\frac{S_T}{S_t}}\left(\frac{S_T}{S_t}\right)^k \mathbb{I}_{\{S_T<\underline{x}S_t\}} \\
= \; & e^{\alpha\underline{x}}\left(\underline{x}\right)^k \mathbb{I}_{\{S_T<\underline{x}S_t\}} - \left[\alpha e^{\alpha\underline{x}}\left(\underline{x}\right)^k + k e^{\alpha\underline{x}}\left(\underline{x}\right)^{k-1}\right]\frac{1}{S_t}\left(\underline{x}S_t - S_T\right)^+ \\
& + \int_0^{\underline{x}S_t}\left[\frac{\alpha^2}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^k + 2\frac{\alpha k}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-1} + \frac{k(k-1)}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-2}\right](K-S_T)^+ dK.
\end{aligned}$$

Taking expectations under the risk-neutral measure yields:

$$\begin{aligned}
& \mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}\right)^k \mathbb{I}_{A_d}\right] \\
= \; & e^{\alpha\underline{x}}\left(\underline{x}\right)^k Prob_t^*\left[S_T<\underline{x}S_t\right] - \left[\alpha e^{\alpha\underline{x}}\left(\underline{x}\right)^k + k e^{\alpha\underline{x}}\left(\underline{x}\right)^{k-1}\right]\frac{R_{f,t\to T}}{S_t}P_{t\to T}\left[\underline{x}S_t\right] \\
& + \frac{R_{f,t\to T}}{S_t^2}\int_0^{\underline{x}S_t} e^{\alpha\frac{K}{S_t}}\left[\alpha^2\left(\frac{K}{S_t}\right)^k + 2\alpha k\left(\frac{K}{S_t}\right)^{k-1} + k(k-1)\left(\frac{K}{S_t}\right)^{k-2}\right]P_{t\to T}[K]\, dK. \quad \text{(IA.32)}
\end{aligned}$$

Equation IA.32 can be combined with Equation IA.27 to compute the risk-neutral moments required for Equation IA.3 in Remark 3 with $\mathbb{I}_{A_s} = \mathbb{I}_{A_d}$.

IA.4.4.3 CARA: Upside Risk-Neutral Moments: $\mathbb{E}_t^*\left[e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}\right)^k \mathbb{I}_{A_u}\right]$

Set the function $h(\cdot)$ in Equation IA.5 to that in Equation IA.28. Next, set $y_0 = \overline{x}S_t$. Evaluating the function and its first derivative at values needed for Equation IA.5 yields:

$$h^{CARA}\left(\overline{x}S_t\right) = e^{\alpha\overline{x}}\left(\overline{x}\right)^k \text{ and}$$

$$h_y^{CARA}\left(\overline{x}S_t\right) = \frac{\alpha}{S_t}e^{\alpha\overline{x}}\left(\overline{x}\right)^k + \frac{k}{S_t}e^{\alpha\overline{x}}\left(\overline{x}\right)^{k-1}.$$

Note that $h_{yy}^{CARA}(K)$ based on Equation IA.30 is unchanged. We can substitute these into the Carr and Madan (2001) spanning formula (Equation IA.5) and multiply the whole equation by $\mathbb{I}_{\{S_T>\overline{x}S_t\}}$ to obtain:

$$\begin{aligned}
& e^{\alpha\frac{S_T}{S_t}}\left(\frac{S_T}{S_t}\right)^k \mathbb{I}_{\{S_T>\overline{x}S_t\}} \\
= \; & e^{\alpha\overline{x}}\left(\overline{x}\right)^k \mathbb{I}_{\{S_T>\overline{x}S_t\}} + \left(S_T-\overline{x}S_t\right)\left[\frac{\alpha}{S_t}e^{\alpha\overline{x}}\left(\overline{x}\right)^k + \frac{k}{S_t}e^{\alpha\overline{x}}\left(\overline{x}\right)^{k-1}\right]\mathbb{I}_{\{S_T>\overline{x}S_t\}} \\
& + \int_0^{\overline{x}S_t}\left[\frac{\alpha^2}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^k + 2\frac{\alpha k}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-1} + \frac{k(k-1)}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-2}\right](K-S_T)^+ \mathbb{I}_{\{S_T>\overline{x}S_t\}} dK \\
& + \int_{\overline{x}S_t}^{\infty}\left[\frac{\alpha^2}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^k + 2\frac{\alpha k}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-1} + \frac{k(k-1)}{S_t^2}e^{\alpha\frac{K}{S_t}}\left(\frac{K}{S_t}\right)^{k-2}\right](S_T-K)^+ \mathbb{I}_{\{S_T>\overline{x}S_t\}} dK.
\end{aligned}$$

Rearranging and simplifying (noting that the first integral is zero) yields:

$$
\begin{aligned}
& e^{\alpha \frac{S_T}{S_t}} \left(\frac{S_T}{S_t}\right)^k \mathbb{I}_{\{S_T > \overline{x} S_t\}} \\
= \quad & e^{\alpha \overline{x}} \left(\overline{x}\right)^k \mathbb{I}_{\{S_T > \overline{x} S_t\}} + \left[\alpha e^{\alpha \overline{x}} \left(\overline{x}\right)^k + k e^{\alpha \overline{x}} \left(\overline{x}\right)^{k-1}\right] \frac{1}{S_t} \left(S_T - \overline{x} S_t\right)^+ \\
& + \frac{1}{S_t^2} \int_{\overline{x} S_t}^{\infty} \left[\alpha^2 e^{\alpha \frac{K}{S_t}} \left(\frac{K}{S_t}\right)^k + 2 \alpha k e^{\alpha \frac{K}{S_t}} \left(\frac{K}{S_t}\right)^{k-1} + k \left(k-1\right) e^{\alpha \frac{K}{S_t}} \left(\frac{K}{S_t}\right)^{k-2}\right] \left(S_T - K\right)^+ dK.
\end{aligned}
$$

Taking expectations under the risk-neutral measure yields:

$$
\begin{aligned}
& \mathbb{E}_t^* \left[e^{\alpha R_{M,t\to T}} \left(R_{M,t\to T}\right)^k \mathbb{I}_{A_u}\right] \\
= \quad & e^{\alpha \overline{x}} \left(\overline{x}\right)^k Prob_t^* \left[S_T > \overline{x} S_t\right] + \left[\alpha e^{\alpha \overline{x}} \left(\overline{x}\right)^k + k e^{\alpha \overline{x}} \left(\overline{x}\right)^{k-1}\right] \frac{R_{f,t\to T}}{S_t} C_{t\to T} \left[\overline{x} S_t\right] \\
& + \frac{R_{f,t\to T}}{S_t^2} \int_{\overline{x} S_t}^{\infty} e^{\alpha \frac{K}{S_t}} \left[\alpha^2 \left(\frac{K}{S_t}\right)^k + 2 \alpha k \left(\frac{K}{S_t}\right)^{k-1} + k \left(k-1\right) \left(\frac{K}{S_t}\right)^{k-2}\right] C_{t\to T} \left[K\right] dK. \quad \text{(IA.33)}
\end{aligned}
$$

Equation IA.33 can be combined with Equation IA.27 to compute the risk-neutral moments required for Equation IA.3 in Remark 3 with $\mathbb{I}_{A_s} = \mathbb{I}_{A_u}$.

IA.4.4.4 CARA: Central Risk-Neutral Moments: $\mathbb{E}_t^* \left[e^{\alpha R_{M,t\to T}} \left(R_{M,t\to T}\right)^k \mathbb{I}_{A_c}\right]$

The identity in Equation IA.14 implies the following identity relating the risk-neutral moments:

$$
\begin{aligned}
\mathbb{E}^* \left[e^{\alpha R_{M,t\to T}} \left(R_{M,t\to T}\right)^k \mathbb{I}_{A_c}\right] \quad \equiv \quad & \mathbb{E}_t^* \left[e^{\alpha R_{M,t\to T}} \left(R_{M,t\to T}\right)^k\right] \\
& - \mathbb{E}^* \left[e^{\alpha R_{M,t\to T}} \left(R_{M,t\to T}\right)^k \mathbb{I}_{A_d}\right] \\
& - \mathbb{E}^* \left[e^{\alpha R_{M,t\to T}} \left(R_{M,t\to T}\right)^k \mathbb{I}_{A_u}\right]
\end{aligned}
$$

where we have made use of the definitions $\mathbb{I}_{A_c} \equiv \mathbb{I}_{\{\underline{x} S_t \le S_T \le \overline{x} S_T\}}$, $\mathbb{I}_{A_d} \equiv \mathbb{I}_{\{S_T < \underline{x} S_t\}}$, and $\mathbb{I}_{A_u} \equiv \mathbb{I}_{\{S_T > \overline{x} S_t\}}$. Substituting in expressions from Equations IA.31, IA.32, and IA.33 and simplifying yields:

$$
\begin{aligned}
& \mathbb{E}^{*}\left[e^{\alpha R_{M,t\to T}}\left(R_{M,t\to T}\right)^{k}\mathbb{I}_{A_c}\right] \\
= \; & e^{\alpha R_{f,t\to T}}\left(R_{f,t\to T}\right)^{k}-e^{\alpha\underline{x}}\left(\underline{x}\right)^{k}Prob_t^{*}\left[S_T<\underline{x}S_t\right]-e^{\alpha\overline{x}}\left(\overline{x}\right)^{k}Prob_t^{*}\left[S_T>\overline{x}S_t\right] \\
& +\left[\alpha e^{\alpha\underline{x}}\left(\underline{x}\right)^{k}+ke^{\alpha\underline{x}}\left(\underline{x}\right)^{k-1}\right]\frac{R_{f,t\to T}}{S_t}P_{t\to T}\left[\underline{x}S_t\right]-\left[\alpha e^{\alpha\overline{x}}\left(\overline{x}\right)^{k}+ke^{\alpha\overline{x}}\left(\overline{x}\right)^{k-1}\right]\frac{R_{f,t\to T}}{S_t}C_{t\to T}\left[\overline{x}S_t\right] \\
& +\frac{R_{f,t\to T}}{S_t^2}\int_{\underline{x}S_t}^{R_{f,t\to T}S_t}e^{\alpha\frac{K}{S_t}}\left[\alpha^2\left(\frac{K}{S_t}\right)^{k}+2\alpha k\left(\frac{K}{S_t}\right)^{k-1}+k\left(k-1\right)\left(\frac{K}{S_t}\right)^{k-2}\right]P_{t\to T}\left[K\right]dK \\
& +\frac{R_{f,t\to T}}{S_t^2}\int_{R_{f,t\to T}S_t}^{\overline{x}S_t}e^{\alpha\frac{K}{S_t}}\left[\alpha^2\left(\frac{K}{S_t}\right)^{k}+2\alpha k\left(\frac{K}{S_t}\right)^{k-1}+k\left(k-1\right)\left(\frac{K}{S_t}\right)^{k-2}\right]C_{t\to T}\left[K\right]dK. \quad \text{(IA.34)}
\end{aligned}
$$

Equation IA.34 can be combined with Equation IA.27 to compute the risk-neutral moments required for Equation IA.3 in Remark 3 with $\mathbb{I}_{A_s}=\mathbb{I}_{A_c}$.

IA.4.5 Risk-Neutral Moments for HARA Utility-Based Physical Moments

Proof. Proof of Remark 4. The conditional truncated moment is

$$
\begin{aligned}
\mathbb{M}_{t\to T}^{(n)}\left[A_s\right] &= \mathbb{E}_t^{*}\left[\frac{\mathbb{E}_t\left[m_{t\to T}\right]}{m_{t\to T}}\left(R_{M,t\to T}-R_{f,t\to T}\right)^{n}\mathbb{I}_{A_s}\right] \\
&= \frac{\mathbb{E}_t^{*}\left[\left(-a^{*}\left(R_{M,t\to T}/R_{f,t\to T}\right)-1\right)^{1-\gamma}\left(R_{M,t\to T}-R_{f,t\to T}\right)^{n}\mathbb{I}_{A_s}\right]}{\mathbb{E}_t^{*}\left[\left(-a^{*}\left(R_{M,t\to T}/R_{f,t\to T}\right)-1\right)^{1-\gamma}\right]}. \quad \text{(IA.35)}
\end{aligned}
$$

■

Next, we show how to compute risk-neutral moments in Remark 4 (Equation IA.4). Specifically, we would like to compute moments of the form $\mathbb{E}_t^{*}\left[\left(-\frac{a^{*}}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}-R_{f,t\to T}\right)^{n}\mathbb{I}_{A_s}\right]$. We first apply the binomial theorem to show:

$$
\begin{aligned}
& \mathbb{E}_t^{*}\left[\left(-\frac{a^{*}}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}-R_{f,t\to T}\right)^{n}\mathbb{I}_{A_s}\right] \\
= \; & \sum_{k=0}^{n}\frac{\left(-1\right)^{n-k}n!}{k!\left(n-k\right)!}\left(R_{f,t\to T}\right)^{n-k}\mathbb{E}_t^{*}\left[\left(-\frac{a^{*}}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}\right)^{k}\mathbb{I}_{A_s}\right]. \quad \text{(IA.36)}
\end{aligned}
$$

So we need only compute moments of $\left(-\frac{a^{*}}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}\right)^{k}\mathbb{I}_{A_s}$ in order to construct moments of $\left(-\frac{a^{*}}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}-R_{f,t\to T}\right)^{n}\mathbb{I}_{A_s}$. We again make use of the Carr and Madan (2001) spanning formula (Equation IA.5) to compute these moments as functions of options prices.

IA.4.5.1 HARA: Untruncated Risk-Neutral Moments:

Our goal is to compute:

$$\mathbb{E}_t^* \left[\left(-\frac{a^*}{R_{f,t\to T}} R_{M,t\to T} - 1 \right)^{1-\gamma} \left(R_{M,t\to T} \right)^k \right]$$

Set the function $h(\cdot)$ in Equation IA.5 to

$$h^{HARA}(S_T) = \left(-\frac{a^*}{R_{f,t\to T}} \left(\frac{S_T}{S_t} \right) - 1 \right)^{1-\gamma} \left(\frac{S_T}{S_t} \right)^k . \tag{IA.37}$$

Derivatives of this function are

$$\begin{aligned} h_y^{HARA}(S_T) &= -(1-\gamma) \frac{a^*}{R_{f,t\to T} S_t} \left(-\frac{a^*}{R_{f,t\to T}} \left(\frac{S_T}{S_t} \right) - 1 \right)^{-\gamma} \left(\frac{S_T}{S_t} \right)^k \\ &\quad + \frac{k}{S_t} \left(-\frac{a^*}{R_{f,t\to T}} \left(\frac{S_T}{S_t} \right) - 1 \right)^{1-\gamma} \left(\frac{S_T}{S_t} \right)^{k-1} \text{ and} \end{aligned} \tag{IA.38}$$

$$\begin{aligned} h_{yy}^{HARA}(S_T) &= -\gamma(1-\gamma) \left(\frac{a^*}{R_{f,t\to T} S_t} \right)^2 \left(-\frac{a^*}{R_{f,t\to T}} \left(\frac{S_T}{S_t} \right) - 1 \right)^{-\gamma-1} \left(\frac{S_T}{S_t} \right)^k \\ &\quad -2 \frac{k(1-\gamma) a^*}{R_{f,t\to T} S_t^2} \left(-\frac{a^*}{R_{f,t\to T}} \left(\frac{S_T}{S_t} \right) - 1 \right)^{-\gamma} \left(\frac{S_T}{S_t} \right)^{k-1} \\ &\quad + \frac{k(k-1)}{S_t^2} \left(-\frac{a^*}{R_{f,t\to T}} \left(\frac{S_T}{S_t} \right) - 1 \right)^{1-\gamma} \left(\frac{S_T}{S_t} \right)^{k-2} . \end{aligned} \tag{IA.39}$$

Next, set $y_0 = R_{f,t\to T} S_t$. Evaluating the function and its derivatives at values needed for Equation IA.5 yields:

$$h^{HARA}(R_{f,t\to T} S_t) = (-a^* - 1)^{1-\gamma} (R_{f,t\to T})^k ,$$

$$\begin{aligned} h_y^{HARA}(R_{f,t\to T} S_t) &= -(1-\gamma) \frac{a^*}{R_{f,t\to T} S_t} (-a^* - 1)^{-\gamma} (R_{f,t\to T})^k \\ &\quad + \frac{k}{S_t} (-a^* - 1)^{1-\gamma} (R_{f,t\to T})^{k-1} , \text{ and} \end{aligned}$$

$$
\begin{aligned}
h_{yy}^{HARA}(K) \;=\; & -\gamma(1-\gamma)\left(\frac{a^*}{R_{f,t\to T}S_t}\right)^2\left(-\frac{a^*}{R_{f,t\to T}}\left(\frac{K}{S_t}\right)-1\right)^{-\gamma-1}\left(\frac{K}{S_t}\right)^k \\
& -2\frac{k(1-\gamma)a^*}{R_{f,t\to T}S_t^2}\left(-\frac{a^*}{R_{f,t\to T}}\left(\frac{K}{S_t}\right)-1\right)^{-\gamma}\left(\frac{K}{S_t}\right)^{k-1} \\
& +\frac{k(k-1)}{S_t^2}\left(-\frac{a^*}{R_{f,t\to T}}\left(\frac{K}{S_t}\right)-1\right)^{1-\gamma}\left(\frac{K}{S_t}\right)^{k-2}.
\end{aligned}
$$

Substituting these expressions into Equation IA.5 yields:

$$
\begin{aligned}
& \left(-\frac{a^*}{R_{f,t\to T}}\left(\frac{S_T}{S_t}\right)-1\right)^{1-\gamma}\left(\frac{S_T}{S_t}\right)^k \\
= \; & (-a^*-1)^{1-\gamma}(R_{f,t\to T})^k \\
& +(S_T-R_{f,t\to T}S_t)\,h_y^{HARA}[R_{f,t\to T}S_t] \\
& +\int_0^{R_{f,t\to T}S_t} h_{yy}^{HARA}(K)(K-S_T)^+\,dK \\
& +\int_{R_{f,t\to T}S_t}^{\infty} h_{yy}^{HARA}(K)(S_T-K)^+\,dK.
\end{aligned}
$$

Taking expectations under the risk-neutral measure at time t yields:

$$
\begin{aligned}
& \mathbb{E}_t^*\left[\left(-\frac{a^*}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}(R_{M,t\to T})^k\right] \\
= \; & (-a^*-1)^{1-\gamma}(R_{f,t\to T})^k \\
& +R_{f,t\to T}\int_0^{R_{f,t\to T}S_t} h_{yy}^{HARA}(K)\,P_{t\to T}[K]\,dK \\
& +R_{f,t\to T}\int_{R_{f,t\to T}S_t}^{\infty} h_{yy}^{HARA}(K)\,C_{t\to T}[K]\,dK.
\end{aligned}
\tag{IA.40}
$$

Equation IA.40 can be combined with Equation IA.36 to compute the risk-neutral moments required for Equation IA.4 in Remark 4 when $\mathbb{I}_{A_s}=1$ (i.e., the untruncated moment case).

IA.4.5.2 HARA: Downside Risk-Neutral Moments: $\mathbb{E}_t^*\left[\left(-\frac{a^*}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}(R_{M,t\to T})^k\,\mathbb{I}_{A_d}\right]$

Set the function $h(\cdot)$ in Equation IA.5 to that in Equation IA.37. Next, set $y_0=\underline{x}S_t$. Evaluating the function and its first derivative at values needed for Equation IA.5 yields:

$$
h^{HARA}(\underline{x}S_t) \;=\; \left(-\frac{a^*}{R_{f,t\to T}}\underline{x}-1\right)^{1-\gamma}(\underline{x})^k \;\text{ and}
$$

$$h_y^{HARA}\left(\underline{x}S_t\right) = -\left(1-\gamma\right)\frac{a^*}{R_{f,t\to T}S_t}\left(-\frac{a^*}{R_{f,t\to T}}\underline{x}-1\right)^{-\gamma}\left(\underline{x}\right)^k + \frac{k}{S_t}\left(-\frac{a^*}{R_{f,t\to T}}\underline{x}-1\right)^{1-\gamma}\left(\underline{x}\right)^{k-1}.$$

Note that $h_{yy}^{HARA}\left(K\right)$ based on Equation IA.39 is unchanged. We can substitute these into the Carr and Madan (2001) spanning formula (Equation IA.5) and multiply the whole equation by $\mathbb{I}_{\{S_T<\underline{x}S_t\}}$ to obtain:

$$\begin{aligned}
&\left(-\frac{a^*}{R_{f,t\to T}}\left(\frac{S_T}{S_t}\right)-1\right)^{1-\gamma}\left(\frac{S_T}{S_t}\right)^k\mathbb{I}_{\{S_T<\underline{x}S_t\}}\\
=\;&\left(-\frac{a^*}{R_{f,t\to T}}\underline{x}-1\right)^{1-\gamma}\left(\underline{x}\right)^k\mathbb{I}_{\{S_T<\underline{x}S_t\}}\\
&+\left(S_T-\underline{x}S_t\right)h_y^{HARA}\left(\underline{x}S_t\right)\mathbb{I}_{\{S_T<\underline{x}S_t\}}\\
&+\int_0^{\underline{x}S_t}h_{yy}^{HARA}\left(K\right)\left(K-S_T\right)^+\mathbb{I}_{\{S_T<\underline{x}S_t\}}dK\\
&+\int_{\underline{x}S_t}^{\infty}h_{yy}^{HARA}\left(K\right)\left(S_T-K\right)^+\mathbb{I}_{\{S_T<\underline{x}S_t\}}dK.
\end{aligned}$$

Rearranging and simplifying (noting that the second integral is zero) yields:

$$\begin{aligned}
&\left(-\frac{a^*}{R_{f,t\to T}}\left(\frac{S_T}{S_t}\right)-1\right)^{1-\gamma}\left(\frac{S_T}{S_t}\right)^k\mathbb{I}_{\{S_T<\underline{x}S_t\}}\\
=\;&\left(-\frac{a^*}{R_{f,t\to T}}\underline{x}-1\right)^{1-\gamma}\left(\underline{x}\right)^k\mathbb{I}_{\{S_T<\underline{x}S_t\}}\\
&-h_y^{HARA}\left(\underline{x}S_t\right)\left(\underline{x}S_t-S_T\right)^+\\
&+\int_0^{\underline{x}S_t}h_{yy}^{HARA}\left(K\right)\left(K-S_T\right)^+dK.
\end{aligned}$$

Taking expectations under the risk-neutral measure yields:

$$\begin{aligned}
&\mathbb{E}_t^*\left[\left(-\frac{a^*}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}\right)^k\mathbb{I}_{A_d}\right]\\
=\;&\left(-\frac{a^*}{R_{f,t\to T}}\underline{x}-1\right)^{1-\gamma}\left(\underline{x}\right)^k Prob_t^*\left[S_T<\underline{x}S_t\right]\\
&-R_{f,t\to T}h_y^{HARA}\left(\underline{x}S_t\right)P_{t\to T}\left[\underline{x}S_t\right]\\
&+R_{f,t\to T}\int_0^{\underline{x}S_t}h_{yy}^{HARA}\left(K\right)P_{t\to T}\left[K\right]dK.
\end{aligned}\tag{IA.41}$$

Equation IA.41 can be combined with Equation IA.36 to compute the risk-neutral moments required for Equation IA.4 in Remark 4 with $\mathbb{I}_{A_s}=\mathbb{I}_{A_d}$.

IA.4.5.3 HARA: Upside Risk-Neutral Moments: $\mathbb{E}_t^*\left[\left(-\frac{a^*}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}\right)^k\mathbb{I}_{A_u}\right]$

Set the function $h(\cdot)$ in Equation IA.5 to that in Equation IA.37. Next, set $y_0 = \overline{x}S_t$. Evaluating the function and its first derivative at values needed for Equation IA.5 yields:

$$h^{HARA}(\overline{x}S_t) = \left(-\frac{a^*}{R_{f,t\to T}}\overline{x}-1\right)^{1-\gamma}(\overline{x})^k \text{ and}$$

$$\begin{aligned} h_y^{HARA}(\overline{x}S_t) &= -(1-\gamma)\frac{a^*}{R_{f,t\to T}S_t}\left(-\frac{a^*}{R_{f,t\to T}}\overline{x}-1\right)^{-\gamma}(\overline{x})^k \\ &\quad +\frac{k}{S_t}\left(-\frac{a^*}{R_{f,t\to T}}\overline{x}-1\right)^{1-\gamma}(\overline{x})^{k-1}. \end{aligned}$$

Note that $h_{yy}^{HARA}(K)$ based on Equation IA.39 is unchanged. We can substitute these into the Carr and Madan (2001) spanning formula (Equation IA.5) and multiply the whole equation by $\mathbb{I}_{\{S_T>\overline{x}S_t\}}$ to obtain:

$$\begin{aligned} &\left(-\frac{a^*}{R_{f,t\to T}}\left(\frac{S_T}{S_t}\right)-1\right)^{1-\gamma}\left(\frac{S_T}{S_t}\right)^k\mathbb{I}_{\{S_T>\overline{x}S_t\}} \\ = &\left(-\frac{a^*}{R_{f,t\to T}}\overline{x}-1\right)^{1-\gamma}(\overline{x})^k\,\mathbb{I}_{\{S_T>\overline{x}S_t\}} \\ &+(S_T-\overline{x}S_t)\,h_y^{HARA}(\overline{x}S_t)\,\mathbb{I}_{\{S_T>\overline{x}S_t\}} \\ &+\int_0^{\overline{x}S_t}h_{yy}^{HARA}(K)(K-S_T)^+\,\mathbb{I}_{\{S_T>\overline{x}S_t\}}dK \\ &+\int_{\overline{x}S_t}^{\infty}h_{yy}^{HARA}(K)(S_T-K)^+\,\mathbb{I}_{\{S_T>\overline{x}S_t\}}dK. \end{aligned}$$

Rearranging and simplifying (noting that the first integral is zero) yields:

$$\begin{aligned} &\left(-\frac{a^*}{R_{f,t\to T}}\left(\frac{S_T}{S_t}\right)-1\right)^{1-\gamma}\left(\frac{S_T}{S_t}\right)^k\mathbb{I}_{\{S_T>\overline{x}S_t\}} \\ = &\left(-\frac{a^*}{R_{f,t\to T}}\overline{x}-1\right)^{1-\gamma}(\overline{x})^k\,\mathbb{I}_{\{S_T>\overline{x}S_t\}} \\ &+h_y^{HARA}(\overline{x}S_t)(S_T-\overline{x}S_t)^+ \\ &+\int_{\overline{x}S_t}^{\infty}h_{yy}^{HARA}(K)(S_T-K)^+\,dK. \end{aligned}$$

Taking expectations under the risk-neutral measure yields:

$$
\begin{aligned}
& \mathbb{E}_t^*\left[\left(-\frac{a^*}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}\right)^k\mathbb{I}_{\{S_T>\overline{x}S_t\}}\right] \\
= \; & \left(-\frac{a^*}{R_{f,t\to T}}\overline{x}-1\right)^{1-\gamma}\left(\overline{x}\right)^k Prob_t^*\left[S_T>\overline{x}S_t\right] \\
& +R_{f,t\to T}h_y^{HARA}\left(\overline{x}S_t\right)C_{t\to T}\left[\overline{x}S_t\right] \\
& +R_{f,t\to T}\int_{\overline{x}S_t}^{\infty}h_{yy}^{HARA}\left(K\right)C_{t\to T}\left[K\right]dK.
\end{aligned}
\tag{IA.42}
$$

Equation IA.42 can be combined with Equation IA.36 to compute the risk-neutral moments required for Equation IA.4 in Remark 4 with $\mathbb{I}_{A_s}=\mathbb{I}_{A_u}$.

IA.4.5.4 HARA: Central Risk-Neutral Moments: $\mathbb{E}_t^*\left[\left(-\frac{a^*}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}\right)^k\mathbb{I}_{A_c}\right]$

The identity in Equation IA.14 implies the following identity relating the risk-neutral moments:

$$
\begin{aligned}
& \mathbb{E}^*\left[\left(-\frac{a^*}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}\right)^k\mathbb{I}_{A_c}\right] \\
\equiv \; & \mathbb{E}_t^*\left[\left(-\frac{a^*}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}\right)^k\right] \\
& -\mathbb{E}^*\left[\left(-\frac{a^*}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}\right)^k\mathbb{I}_{A_d}\right] \\
& -\mathbb{E}^*\left[\left(-\frac{a^*}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}\right)^k\mathbb{I}_{A_u}\right]
\end{aligned}
$$

where we have made use of the definitions $\mathbb{I}_{A_c}\equiv\mathbb{I}_{\{\underline{x}S_t\leq S_T\leq\overline{x}S_T\}}$, $\mathbb{I}_{A_d}\equiv\mathbb{I}_{\{S_T<\underline{x}S_t\}}$, and $\mathbb{I}_{A_u}\equiv\mathbb{I}_{\{S_T>\overline{x}S_t\}}$. Substituting in expressions from Equations IA.40, IA.41, and IA.42 and simplifying yields:

$$
\begin{aligned}
& \mathbb{E}^*\left[\left(-\frac{a^*}{R_{f,t\to T}}R_{M,t\to T}-1\right)^{1-\gamma}\left(R_{M,t\to T}\right)^k\mathbb{I}_{A_c}\right] \\
= \; & \left(-a^*-1\right)^{1-\gamma}\left(R_{f,t\to T}\right)^k \\
& -\left(-\frac{a^*}{R_{f,t\to T}}\underline{x}-1\right)^{1-\gamma}\left(\underline{x}\right)^k Prob_t^*\left[S_T<\underline{x}S_t\right]-\left(-\frac{a^*}{R_{f,t\to T}}\overline{x}-1\right)^{1-\gamma}\left(\overline{x}\right)^k Prob_t^*\left[S_T>\overline{x}S_t\right] \\
& +R_{f,t\to T}\left(h_y^{HARA}\left(\underline{x}S_t\right)P_{t\to T}\left[\underline{x}S_t\right]-h_y^{HARA}\left(\overline{x}S_t\right)C_{t\to T}\left[\overline{x}S_t\right]\right) \\
& +R_{f,t\to T}\left(\int_{\underline{x}S_t}^{R_{f,t\to T}S_t}h_{yy}^{HARA}\left(K\right)P_{t\to T}\left[K\right]dK+\int_{R_{f,t\to T}S_t}^{\overline{x}S_t}h_{yy}^{HARA}\left(K\right)C_{t\to T}\left[K\right]dK\right).
\end{aligned}
\tag{IA.43}
$$

Equation IA.43 can be combined with Equation IA.36 to compute the risk-neutral moments required for Equation IA.4 in Remark 4 with $\mathbb{I}_{A_s} = \mathbb{I}_{A_c}$.

IA.5 Nonlinear Least Squares Estimation of Preference Parameters

We would like to estimate the preference parameters $\tau(x_s)$, $\rho(x_s)$, and $\kappa(x_s)$ at three points in the market return space corresponding to $s \in \{d, c, u\}$. These preference parameters are required to compute the physical moments (via Corollary 1)[40] needed to implement the risk premium decomposition in Proposition 3. We use the relationship between physical and risk-neutral moments from Corollary 1 to estimate preference parameters. Start by writing powers of realized excess market returns as:

$$(R_{M,t\to T} - R_{f,t\to T})^n \;=\; \mathbb{E}_t\left[(R_{M,t\to T} - R_{f,t\to T})^n\right] + \varepsilon^{(n)}_{t\to T,s}$$

Applying Corollary 1, we can replace $\mathbb{E}_t\left[(R_{M,t\to T} - R_{f,t\to T})^n\right]$ to obtain:

$$(R_{M,t\to T} - R_{f,t\to T})^n \;=\; \mathbb{M}^{*(n)}_{t\to T} + \frac{\sum_{k=1}^{\infty}\sum_{j=0}^{k}\lambda_t(x_s,k,j)\left(\mathbb{M}^{*(n+k-j)}_{t\to T} - \mathbb{M}^{*(k-j)}_{t\to T}\mathbb{M}^{*(n)}_{t\to T}\right)}{1+\sum_{k=1}^{\infty}\sum_{j=0}^{k}\lambda_t(x_s,k,j)\,\mathbb{M}^{*(k-j)}_{t\to T}} + \varepsilon^{(n)}_{t\to T,s}.$$

We truncate this summation at $k = 3$ for tractability. This balances truncation error induced by choosing lower k with the fact that estimated higher-order risk-neutral moments (needed for higher k) can be inaccurate.[41] Assuming the truncation error is time invariant, we can write:

$$(R_{M,t\to T} - R_{f,t\to T})^n = a^{(n)}_{T,s} + \mathbb{M}^{*(n)}_{t\to T} + \frac{\sum_{k=1}^{3}\sum_{j=0}^{k}\lambda_t(x_s,k,j)\left(\mathbb{M}^{*(n+k-j)}_{t\to T} - \mathbb{M}^{*(k-j)}_{t\to T}\mathbb{M}^{*(n)}_{t\to T}\right)}{1+\sum_{k=1}^{3}\sum_{j=0}^{k}\lambda_t(x_s,k,j)\,\mathbb{M}^{*(k-j)}_{t\to T}} + \varepsilon^{(n)}_{t\to T,s} \quad \text{(IA.44)}$$

where we include the constant $a^{(n)}_{T,s}$ to account for the truncation error induced by limiting the upper limit on the sum over k to be 3.[42] To the extent that this error may be time

[40]Recall that this relationship is based on Corollary1, $\mathbb{M}^{*(n)}_{t\to T}$ are untruncated risk-neutral moments, and $\lambda_t(x_s,k,j)$ is a function of the preference parameters $\tau(x_s)$, $\rho(x_s)$, and $\kappa(x_s)$ according to Equations 26 and 12.

[41]For example, see Rompolis and Tzavalis (2017), who investigate the accuracy of higher-order risk-neutral moments computed from options when the cross-section of options is limited.

[42]Note that all moments in Equation IA.44 are untruncated moments (i.e., $\mathbb{M}^{*(n)}_{t\to T}[A]$). We suppress the $[A]$ dependence here for simplicity.

varying, it will be relegated to the error term, $\varepsilon_{t\to T,\,s}^{(n)}$. Note that this can be applied to moments of any order (n), any time horizon (T). Recall that $\lambda_t\left(x_s,k,j\right)$ is a function of the preference parameters $\tau\left(x_s\right)$, $\rho\left(x_s\right)$, and $\kappa\left(x_s\right)$ according to Equations 26 and 12. We set $\underline{x} = 0.9$ and $\overline{x} = 1.1$ with $x_d = 0.85$, $x_c = 1$, and $x_d = 1.15$ in all reported results. That is, we are interested in studying risk premia associated with down market returns less than -10%, central market returns between -10% and +10%, and up market returns greater than +10%.

Note that we must estimate nine total preference parameters for each horizon of interest: three parameters ($\tau\left(x_s\right)$, $\rho\left(x_s\right)$, and $\kappa\left(x_s\right)$) for each of the three regions of interest ($s \in \{d,c,u\}$) in the return space. Given realized excess market returns and measured risk-neutral moments, we estimate the preference parameters $\tau\left(x_s\right)$, $\rho\left(x_s\right)$, and $\kappa\left(x_s\right)$ using non-linear weighted least squares to minimize the squared error using three versions of Equation IA.44: $n = 1$, 2, and 3. Each value of n brings with it three equations: one for each set of preference parameters in each of our three regions of interest. Therefore, we have nine total equations and sets of error terms that are generated using Equation IA.44 in our estimation.

Since we are also interested in estimating truncated risk premia, we need to ensure that the preference parameters satisfy the restriction that the sum of truncated physical moments equals the untruncated physical moment (at least on average across time). This restriction can be introduced to the nonlinear least squares estimation by considering relationships of the form

$$\begin{aligned}
& \left(R_{M,t\to T} - R_{f,t\to T}\right)^n \\
= \;& a_T^{(n)} + \\
& \sum_{s\in\{d,c,u\}} \left[\mathbb{M}_{t\to T}^{*(n)}\left[A_s\right] + \frac{\sum_{k=1}^{3}\sum_{j=0}^{k} \lambda_t\left(x_s,k,j\right)\left(\mathbb{M}_{t\to T}^{*(n+k-j)}\left[A_s\right] - \mathbb{M}_{t\to T}^{*(k-j)}\left[A\right]\mathbb{M}_{t\to T}^{*(n)}\left[A_s\right]\right)}{1+\sum_{k=1}^{\infty}\sum_{j=0}^{k}\lambda_t\left(x_s,k,j\right)\mathbb{M}_{t\to T}^{*(k-j)}\left[A\right]} \right] + \varepsilon_{t\to T}^{(n)}.
\end{aligned} \tag{IA.45}$$

This relationship follows from Corollary 1 and the identity in Equation 28. We allow for the relationship to not hold exactly both conditionally (through inclusion of $\varepsilon_{t\to T}^{(n)}$) and on average across time (through the inclusion of $a_T^{(n)}$). We do not require the relationship to hold exactly conditionally due to the use of slightly different data from options prices when computing truncated moments relative to untruncated moments (see Internet Appendix IA.4.1.1 for a description of how these moments are computed using option price data). Given our estimated parameter values, though, we also find that this restriction holds well both conditionally and unconditionally (i.e., $a_T^{(n)} \approx 0$ and the magnitude of $\varepsilon_{t\to T}^{(n)}$ are small relative to the magnitude of the estimated physical moments, $\mathbb{M}_{t\to T}^{(n)} \equiv \mathbb{E}_t\left[\left(R_{M,t\to T} - R_{f,t\to T}\right)^n\right]$). With $n = 1$, 2, and 3, Equation IA.45 adds three additional equations and sets of error terms to include in the estimation.[43]

When minimizing the sum of squared errors implied by Equations IA.44 and IA.45, we weight the error terms by the inverse standard deviation of the left-hand-side time series associated with each equation (i.e. the inverse of the standard deviation of $(R_{M,t\to T} - R_{f,t\to T})^n$). We do this because the volatilities of each left-hand-side variable are naturally of a different magnitude for different n values. We would like the error terms associated with equations having different n values to have approximately the same weight in the least squares minimization, which is effectively achieved using this weighting scheme.

The final ingredient in our estimation comes in the form of a Ridge-type penalty on the estimated preference parameters. That is, in addition to the sum of squared error terms from the twelve sets of restrictions discussed above, we add an additional term of the form

$$\tilde{\varepsilon}_T^2 = \phi_T \sum_{s\in\{d,c,u\}} \left[\tau_T\left(x_s\right)^2 + \rho_T\left(x_s\right)^2 + \kappa_T\left(x_s\right)^2\right] \tag{IA.46}$$

where ϕ_T is a tuning parameter. We add this penalty to the squared error objective function for estimations at each horizon, T. We include the parameter horizon dependence here explicitly for clarity. We include the penalty term since the right-hand-side risk-neutral moments are highly correlated. This can induce excessive noise in the preference parameter estimates (see Hastie, Tibshirani, and Friedman, 2009, pp. 63-64). Adding a small penalty of this form can help reduce variance in the estimated parameters without increasing estimation bias much. We select the tuning parameter, ϕ_T, using a standard approach. We apply a 10-fold cross validation to the estimation and find that the test error is approximately flat for tuning parameter values below 10^{-3}. We select a moderate value of $\phi_T = 2\times 10^{-5}$ across all horizons to mitigate estimation bias induced by the penalty.

Given these ingredients, we estimate preference parameters separately at each horizon of interest (30, 60, 90, 180, and 360 days) by minimizing an objective function that sums squared errors from the 12 sets of equations described above (Equations IA.44 and IA.45) and the penalty term in Equation IA.46 using daily data from January, 1996 through June, 2019.

IA.6 Projection of Generic SDF onto Aggregate Wealth

Without loss of generality, we set $T = t+1$ to be consistent with the notation in the representative agent models. Denote by $M_{t\to t+1}$ the projection of the representative agent SDF

[43] We could also include equation restrictions related to individual truncated moments (i.e., expressions for $\mathbb{M}^{(n)}_{t\to T}[A_s]$ according to Corollary 1 and using $(R_{M,t\to T} - R_{f,t\to T})^n\,\mathbb{I}_{A_s}$ as realizations in the left hand side of Equation IA.45 (without the summation). We choose not to use these since there are not many instances where $\mathbb{I}_{A_d} = 1$ nor are there many instances of $\mathbb{I}_{A_u} = 1$ in the data (i.e., the extreme market events for which we are interested in estimating risk premia) are not often realized in the data.

on a set spanned by aggregate wealth $\{1, W_{t+1}, .W_{t+1}^2, .W_{t+1}^3 ...\}$. The projected SDF, which can be written as $M_{t\to t+1} = h[W_{t+1}]$, can alternatively be expressed as

$$M_{t\to t+1} = h(W_{ct}R_{M,t\to t+1}),$$

where $W_{ct} = W_t - C_t$ and $R_{M,t\to t+1}$ is a proxy for the market return. This projected SDF can alternatively be written as

$$M_{t\to t+1} = \mu_t g(W_{ct}R_{M,t\to t+1}),$$

where μ_t is a constant. Here, the function $g[.]$ is defined as $g[x] = (\mu_t)^{-1} h[W_{ct}x]$. Since the projected SDF correctly prices any contingent claim whose payoff depends only on the market return, the constant μ_t can be written as

$$\mu_t = M_{t\to T}(g(W_{ct}R_{M,t\to t+1}))^{-1}$$

where μ_t is a constant equal to its own expected value. This enables us to write,

$$\begin{aligned}\mu_t &= (\mathbb{E}_t[M_{t\to t+1}])\,\mathbb{E}_t\left[\frac{M_{t\to t+1}}{\mathbb{E}_t[M_{t\to t+1}]}(g(W_{ct}R_{M,t\to t+1}))^{-1}\right] \\ &= (\mathbb{E}_t[M_{t\to t+1}])\,\mathbb{E}_t^*\left[(g(W_{ct}R_{M,t\to t+1}))^{-1}\right].\end{aligned}$$

Thus,

$$\frac{\mathbb{E}_t[M_{t\to t+1}]}{M_{t\to t+1}} = \frac{\frac{1}{g(W_{ct}R_{M,t\to t+1})}}{\mathbb{E}_t^*\left[\frac{1}{g(W_{ct}R_{M,t\to t+1})}\right]}. \tag{IA.47}$$

As seen above, the inverse of the SDF has a similar functional form when compared to the inverse of the SDF in Section 1.1. Note that for the projected SDF correctly prices the market return and any contingent claim whose payoff is a function of the market return. Thus, the results in Section 1.1 hold. The functional form of the inverse SDF in equation IA.47 is the same as that used to construct our data-implied decomposition (see Equation 3). However, given a particular representative agent model, the analogous coefficients, $\theta_k(\cdot)$, that come out of a Taylor series expansion of the inverse SDF will be pinned down by the assumptions and parameters in the given model.

IA.7 Results and Proofs Related to Representative Agent Models

Our goal in this section is to derive relationships between state variables and asset pricing moments in each model to allow us to extract state variables at each date, and to compute the model-implied risk premia implied by each model given state variables. The state variable extraction procedure is done as described in the paper with related results necessary for the extraction below. Summary statistics for extracted state variables are provided in Table IA.5.

The risk premia are the same as we have defined in our main draft, $\mathbb{RP}^{(n)}_{t\to T}[A_s]$ (see Equation 29). We therefore need to compute the model-implied physical and risk-neutral moments as defined in our main draft, $\mathbb{M}^{(n)}_{t\to T}[A_s]$ and $\mathbb{M}^{*(n)}_{t\to T}[A_s]$ (see Equations 20 and 21). Note that in many cases it will be easier to compute non-central market return moments rather than excess market return moments as required in our definition of $\mathbb{RP}^{(n)}_{t\to T}[A_s]$. In these cases, we can use the binomial theorem to transform non-centered moments of the market return to excess return moments according to

$$\begin{aligned}
\mathbb{E}_t\left[(R_{M,t\to t+1}-R_{f,t\to t+1})^n\right] &= \sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\left(\mathbb{E}_t\left[R^k_{M,t\to t+1}\right]\right)(-1)^{n-k}R^{n-k}_{f,t\to t+1},\\
\mathbb{E}_t\left[(R_{M,t\to t+1}-R_{f,t\to t+1})^n\,\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] &= \sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\left(\mathbb{E}_t\left[R^k_{M,t\to t+1}\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right]\right)(-1)^{Rn-k}R^{n-k}_{f,t\to t+1}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^*_t\left[(R_{M,t\to t+1}-R_{f,t\to t+1})^n\right] &= \sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\left(\mathbb{E}^*_t\left[R^k_{M,t\to t+1}\right]\right)(-1)^{n-k}R^{n-k}_{f,t\to t+1},\\
\mathbb{E}^*_t\left[(R_{M,t\to t+1}-R_{f,t\to t+1})^n\,\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] &= \sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\left(\mathbb{E}^*_t\left[R^k_{M,t\to t+1}\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right]\right)(-1)^{n-k}R^{n-k}_{f,t\to t+1}.
\end{aligned}$$

Results needed to compute these moments given model state variables are provided below.

IA.7.1 Risk-Neutral Moments when the SDF and Returns are Log-Normally Distributed

The following analysis will be useful for deriving some results related to representative agent models. Without loss of generality, set $T = t + 1$. Note that

$$\begin{aligned}
\log\left(\mathbb{E}^*_t\left(R^n_{t\to t+1}\right)\right) &= \log\left(\mathbb{E}_t\left(\tfrac{M_{t\to t+1}}{\mathbb{E}_t M_{t\to t+1}}R^n_{t\to t+1}\right)\right)\\
&= \log R_{f,t\to t+1} + \log\left(\mathbb{E}_t M_{t\to t+1}R^n_{t\to t+1}\right)\\
&= \log R_{f,t\to t+1} + \mathbb{E}_t\log\left(M_{t\to t+1}\right) + n\mathbb{E}_t\log\left(R_{t\to t+1}\right) + \tfrac{1}{2}\mathbb{VAR}_t\left(\log M_{t\to t+1}\right)\\
&\quad + \tfrac{n^2}{2}VAR_t\log R_{t\to t+1} + n\mathbb{COV}_t\left(\log M_{t\to t+1},\log R_{t\to t+1}\right)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\log\left(\mathbb{E}^*_t\left(R^n_{t\to t+1}\right)\right) &= \log R_{f,t\to t+1} + \mathbb{E}_t\log\left(M_{t\to t+1}\right) + \tfrac{1}{2}\mathbb{VAR}_t\left(\log M_{t\to t+1}\right) + \tfrac{n^2}{2}\mathbb{VAR}_t\log R_{t\to t+1}\\
&\quad + n(\mathbb{E}_t\log\left(R_{t\to t+1}\right) + \mathbb{COV}_t\left(\log M_{t\to t+1},\log R_{t\to t+1}\right)).
\end{aligned}$$

The Euler equation $\mathbb{E}_t M_{t\to t+1} = \frac{1}{R_{f,t\to t+1}}$ allows us to write

$$-\log R_{f,t\to t+1} = \mathbb{E}_t \log\left(M_{t\to t+1}\right) + \frac{1}{2}\mathbb{VAR}_t\left(\log M_{t\to t+1}\right)$$

Thus,

$$\log\left(\mathbb{E}_t^*\left(R_{t\to t+1}^n\right)\right) = n(\mathbb{E}_t \log\left(R_{t\to t+1}\right) + \mathbb{COV}_t\left(\log M_{t\to t+1}, \log R_{t\to t+1}\right)) + \frac{n^2}{2}\mathbb{VAR}_t \log R_{t\to t+1} \quad .$$

Under the log-normality assumption, the Euler equation $\mathbb{E}_t M_{t\to t+1} R_{t\to t+1} = 1$ can be expressed as

$$\begin{aligned} &\mathbb{E}_t \log\left(M_{t\to t+1}\right) + \mathbb{E}_t \log\left(R_{t\to t+1}\right) + \tfrac{1}{2}\mathbb{VAR}_t \log R_{t\to t+1} \\ &+\tfrac{1}{2}\mathbb{VAR}_t\left(\log M_{t\to t+1}\right) + \mathbb{COV}_t\left(\log M_{t\to t+1}, \log R_{t\to t+1}\right) = 0. \end{aligned}$$

The above expression is equivalent to

$$-\log R_{f,t\to t+1} \quad +\mathbb{E}_t \log\left(R_{t\to t+1}\right) + \tfrac{1}{2}\mathbb{VAR}_t \log R_{t\to t+1} + \mathbb{COV}_t\left(\log M_{t\to t+1}, \log R_{t\to t+1}\right) = \quad 0$$

which simplifies to

$$\mathbb{E}_t \log\left(R_{t\to t+1}\right) + \mathbb{COV}_t\left(\log M_{t\to t+1}, \log R_{t\to t+1}\right) = \quad \log R_{f,t\to t+1} - \tfrac{1}{2}\mathbb{VAR}_t \log R_{t\to t+1}.$$

Finally

$$\log\left(\mathbb{E}_t^*\left(R_{t\to t+1}^n\right)\right) = n\log R_{f,t\to t+1} + \frac{n(n-1)}{2}\mathbb{VAR}_t \log R_{t\to t+1} \quad . \tag{IA.48}$$

IA.7.2 Truncated Moments of a Log Normal Distribution

The following lemma will be useful for deriving some results related to representative agent models.

Lemma IA.1. *Assume that a random variable* $\log X$ *follows a normal distribution, from Lien (1985), it follows that*

$$\mathbb{E}[X\mathbb{I}_{X>a}]{=}(\mathcal{N}\left[d_1\right])\exp\left(\mathbb{E}\left(\log X\right) + \frac{1}{2}\mathbb{VAR}\left(\log X\right)\right)$$

with

$$d_1 = \frac{\mathbb{VAR}\left(\log X\right) + \mathbb{E}\left(\log X\right) - \log a}{\sqrt{\mathbb{VAR}\left(\log X\right)}}.$$

and

$$\overline{d}_2 = \overline{d}_1 - \sqrt{\mathbb{VAR}_t\left[\log X\right]}.$$

IA.7.3 Long-Run Risk Models

Solving the representative agent problem via indirect utility, Epstein and Zin (1989) show that the SDF has the form

$$M_{t\to t+1} = \delta^{\theta}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} R_{a,t\to t+1}^{-(1-\theta)}, \tag{IA.49}$$

where C_{t+1} is the consumption level, δ is the time discount rate, and $R_{a,t\to t+1}$ is the gross return on aggregate consumption. The parameter $\theta = (1-\gamma)/\left(1-\frac{1}{\psi}\right)$ where γ is the risk aversion parameter and ψ is the intertemporal elasticity of substitution (IES). This is the utility specification used by all models in this subsection.

IA.7.3.1 Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012)

The economies in both Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012) can be described by the following time series

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \phi_c \sigma_t \eta_{c,t+1}, \\
x_{t+1} &= \rho x_t + \phi_x \sigma_t \eta_{x,t+1}, \\
\sigma_{t+1}^2 &= \overline{\sigma}^2(1-\nu) + \nu\sigma_t^2 + \phi_\sigma \omega_{t+1}, \\
\Delta d_{t+1} &= \mu_d + \phi x_t + \phi_d \sigma_t \eta_{d,t+1} + \phi_{d,c}\sigma_t \eta_{c,t+1},
\end{aligned} \tag{IA.50}$$

where $\eta_{c,t+1}$, $\eta_{x,t+1}$, $\eta_{d,t+1}$, and ω_{t+1} are i.i.d, $\Delta c_{t+1} = \log\frac{C_{t+1}}{C_t}$ is the log consumption growth, and $\Delta d_{t+1} = \log\frac{D_{t+1}}{D_t}$ is the log dividend growth. The Bansal and Yaron (2004) model obtains when $\phi_{d,c} = 0$. In both models, σ_t^2 drives uncertainty in the economies. There are two state variables in each model: x_t and σ_t^2. Given this setup, we show the following results.

Main Results

Result IA.1. *Given the state variables x_t and σ_t^2, the Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012) model-implied log price-dividend ratio is given by*

$$\log\frac{P_t}{D_t} = A_{0,m} + A_{1,m}x_t + A_{2,m}\sigma_t^2, \tag{IA.51}$$

and the risk-neutral market return variance is given by

$$\mathbb{M}_{t\to t+1}^{*(2)}[A] = \exp\left(\mathbb{A}_0^{sq} + \mathbb{A}_1^{sq}x_t + \mathbb{A}_2^{sq}\sigma_t^2\right) - \exp\left(2\mathbb{A}_0^{rf} + 2\mathbb{A}_1^{rf}x_t + 2\mathbb{A}_2^{rf}\sigma_t^2\right). \tag{IA.52}$$

The coefficients $A_{0,m}$, $A_{1,m}$, $A_{2,m}$, $\mathbb{A}_0^{sq}$, $\mathbb{A}_1^{sq}$, and $\mathbb{A}_2^{sq}$ are defined below.

Proof. See below. ■

Result IA.2. *The conditional non-central physical moment and non-central truncated physical moments of the market return are*

$$\begin{aligned}\mathbb{E}_t\left[R^n_{M,t\to t+1}\right] &= \exp\left\{n\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] + \frac{n^2}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right\},\\ \mathbb{E}_t\left[R^n_{M,t\to t+1}\mathbb{I}_{\{R_{M,t+1}>a\}}\right] &= \mathcal{N}\left[\overline{d}_{1,n}\right]\exp\left\{n\left(\mathbb{E}_t\left[\log R_{M,t\to t+1}\right]\right) + \frac{n^2}{2}\left(\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right)\right\},\end{aligned}$$

and

$$\overline{d}_{1,n} = \frac{n^2\left(\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right) + n\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] - n\log a}{n\sqrt{\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]}},$$

where

$$\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] = \mathbb{A}^{er}_0 + \mathbb{A}^{er}_1 x_t + \mathbb{A}^{er}_2\sigma^2_t \text{ and } \mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right] = \mathbb{A}^{vr}_0 + \mathbb{A}^{vr}_1\sigma^2_t$$

Further

$$\mathcal{N}\left[\overline{d}_{2,n}\right] = \mathbb{P}_t\left[n\log R_{M,t\to t+1} > n\log a\right]$$

where

$$\overline{d}_{2,n} = \overline{d}_{1,n} - n\sqrt{\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]}.$$

All parameters are defined below.

Proof. See below. ■

Result IA.3. *The conditional non-central moment and truncated non-central moment of the market return under the risk neutral measure are*

$$\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\right] = \exp\left(n\log R_{f,t\to t+1} + \frac{n(n-1)}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right).$$

$$\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] = \mathcal{N}\left[\overline{d}^*_{1,n}\right]\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\right],$$

where

$$\overline{d}^*_{1,n} = \frac{\mathbb{A}^{'}_{n,t}\mathbb{A}_{n,t} - \log a^*_n}{\sqrt{\mathbb{A}^{'}_{n,t}\mathbb{A}_{n,t}}},$$

and

$$\log a^*_n = \mathbb{A}^{'}_{n,t}\mathbb{A}^R_{n,t}\left(\mathbb{A}^{R'}_{n,t}\mathbb{A}^R_{n,t}\right)^{-1}\left(n\log a - \mu^R_{n,t}\right)$$

with

$$\begin{aligned}\mu^{x,\sigma}_{n,t} &= \mathbb{A}_{0,n} + \mathbb{A}_{1,n}x_t + \mathbb{A}_{2,n}\sigma^2_t,\\ \mathbb{A}^{'}_{n,t} &= \left[\mathbb{A}_{3,n}\sigma_t,\ \mathbb{A}_{4,n}\sigma_t,\ \mathbb{A}_{5,n},\ \mathbb{A}_{6,n}\sigma_t\right],\\ \mu^R_{n,t} &= n\mathbb{A}^{er}_0 + n\mathbb{A}^{er}_1 x_t + n\mathbb{A}^{er}_2\sigma^2_t,\\ \mathbb{A}^{R'}_{n,t} &= \left[n\kappa_{1,m}A_{1,m}\phi_x\sigma_t,\ n\phi_{d,c}\sigma_t,\ n\kappa_{1,m}A_{2,m}\phi_\sigma,\ n\phi_d\sigma_t\right].\end{aligned}$$

Further,

$$\overline{d}^*_{2,n} = \overline{d}^*_{1,n} - \sqrt{\mathbb{A}'_{n,t}\mathbb{A}_{n,t}}$$

and

$$\mathbb{E}^*_t\left[\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] = N\left[\overline{d}^*_{2,n}\right].$$

All parameters are defined below.

Proof. See below. ■

Derivations and Proofs We can use the Campbell and Shiller (1988) approximation to write the log gross return as

$$\log R_{a,t\to t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1},$$

where z is the log price-consumption ratio and

$$\kappa_0 = \log\left(1+e^{\overline{z}}\right) - \kappa_1\overline{z} \text{ and } \kappa_1 = \frac{e^{\overline{z}}}{1+e^{\overline{z}}}.$$

The log price consumption ratio follows

$$z_t = A_0 + A_1 x_t + A_2\sigma_t^2,$$

where

$$A_0 = \frac{\left(\log\delta + \left(1-\frac{1}{\psi}\right)\mu_c + \kappa_0 + \kappa_1 A_2\overline{\sigma}^2(1-\nu) + \frac{1}{2}\theta\left(\kappa_1 A_2\phi_\sigma\right)^2\right)}{(1-\kappa_1)}, \tag{IA.53}$$

$$A_1 = \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho}, \tag{IA.54}$$

$$A_2 = \frac{1}{2}\frac{\theta\left(1-\frac{1}{\psi}\right)^2\phi_c^2 + \theta\left(\kappa_1 A_1\phi_x\right)^2}{1-\kappa_1\nu}. \tag{IA.55}$$

The proof of these coefficients A_0, A_1, and A_2 are given below. Using the Campbell and Shiller (1988) approximation, the log market return can be written as

$$\log R_{M,t\to t+1} = \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + \Delta d_{t+1}, \tag{IA.56}$$

where $r_{t+1} = \log R_{M,t\to t+1}$, $z_{m,t} = \log\left(S_t/D_t\right)$ is the log price-dividend ratio and d_{t+1} is the dividend growth. The price dividend ratio is

$$z_{m,t} = A_{0,m} + A_{1,m}x_t + A_{2,m}\sigma_t^2. \tag{IA.57}$$

where

$$A_{0,m} = \frac{1}{(1-\kappa_{1,m})}\left\{\begin{array}{c} \mu_d + \theta \log\delta - (1-\theta)\kappa_0 + \kappa_{0,m} + (1-\theta) A_0 (1-\kappa_1) + \left(-\frac{\theta}{\psi} - (1-\theta)\right)\mu_c \\ + \{-(1-\theta)\kappa_1 A_2 + \kappa_{1,m} A_{2,m}\}\overline{\sigma}^2 (1-\nu) + \frac{1}{2}(-(1-\theta)\kappa_1 A_2 + \kappa_{1,m}A_{2,m})^2 \phi_\sigma^2 \end{array}\right\} \quad \text{(IA.58)}$$

$$A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1-\kappa_{1,m}\rho}, \quad \text{(IA.59)}$$

$$A_{2,m} = \frac{1}{(1-\kappa_{1,m}\nu)}\left\{\begin{array}{c} \frac{1}{2}\left(\phi_{d,c} - \frac{\theta}{\psi}\phi_c - (1-\theta)\phi_c\right)^2 + \frac{1}{2}(\kappa_{1,m}A_{1,m} - (1-\theta)\kappa_1 A_1)^2 \phi_x^2 \\ + (1-\theta) A_2 (1-\kappa_1\nu) + \frac{1}{2}\phi_d^2 \end{array}\right\}. \quad \text{(IA.60)}$$

We also show that the log risk-free return is

$$\log R_{f,t} = \mathbb{A}_0^{\mathrm{rf}} + \mathbb{A}_1^{\mathrm{rf}} x_t + \mathbb{A}_2^{\mathrm{rf}} \sigma_t^2 \text{ with }, \quad \text{(IA.61)}$$

$$\mathbb{A}_0^{\mathrm{rf}} = -\left\{\begin{array}{c} \theta\log\delta - \gamma\mu_c - (1-\theta)\kappa_0 + (1-\kappa_1)(1-\theta)A_0 \\ -(1-\theta)\kappa_1 A_2 \overline{\sigma}^2 (1-\nu) + \frac{1}{2}(1-\theta)^2 \kappa_1^2 A_2^2 \phi_\sigma^2 \end{array}\right\}, \quad \text{(IA.62)}$$

$$\mathbb{A}_1^{\mathrm{rf}} = -\{-\gamma + (1-\kappa_1\rho)(1-\theta)A_1\}, \quad \text{(IA.63)}$$

$$\mathbb{A}_2^{\mathrm{rf}} = -\left\{(1-\kappa_1\nu)(1-\theta)A_2 + \frac{1}{2}\gamma^2\phi_c^2 + \frac{1}{2}(1-\theta)^2\kappa_1^2 A_1^2 \phi_x^2\right\}. \quad \text{(IA.64)}$$

where the proof of $\mathbb{A}_0^{\mathrm{rf}}$, $\mathbb{A}_1^{\mathrm{rf}}$, and $\mathbb{A}_2^{\mathrm{rf}}$ are is given below:

Proof of the A Coefficients. Observe that

$$\begin{aligned}
& \log M_{t+1} + \log R_{a,t+1} \\
= & \left(\theta\log\delta - \frac{\theta}{\psi}\Delta c_{t+1} - (1-\theta)\log R_{a,t+1}\right) + \log R_{a,t+1} \\
= & \theta\log\delta + \left(\theta - \frac{\theta}{\psi}\right)\Delta c_{t+1} + (\theta\kappa_0 + \theta\kappa_1 z_{t+1} - \theta z_t) \\
= & \left\{\theta\log\delta + \left(\theta - \frac{\theta}{\psi}\right)\mu_c + \theta\kappa_0 + \theta\kappa_1 A_0 + \theta\kappa_1 A_2 \overline{\sigma}^2 (1-\nu) - \theta A_0\right\} \\
& + \left\{\left(\theta - \frac{\theta}{\psi}\right) + \theta\kappa_1 A_1 \rho - \theta A_1\right\} x_t + \left(\theta - \frac{\theta}{\psi}\right)\phi_c \sigma_t \eta_{c,t+1} \\
& + \theta\kappa_1 A_1 \phi_x \sigma_t \eta_{x,t+1} + \{\theta\kappa_1 A_2 \nu - \theta A_2\}\sigma_t^2 + \theta\kappa_1 A_2 \phi_\sigma \omega_{t+1}.
\end{aligned}$$

Therefore, the log of the expected value of $M_{t\to t+1}R_{a,t\to t+1}$ is

$$\begin{aligned}
\log\left(\mathbb{E}\left[M_{t\to t+1}R_{a,t\to t+1}\right]\right) &= \left\{\ \theta\log\delta+\left(\theta-\tfrac{\theta}{\psi}\right)\mu_c+\theta\kappa_0+\theta\kappa_1A_0+\theta\kappa_1A_2\overline{\sigma}^2\left(1-\nu\right)-\theta A_0+\tfrac{1}{2}\left(\theta\kappa_1A_2\phi_\sigma\right)^2\ \right\} \\
&\quad+\left\{\left(\theta-\frac{\theta}{\psi}\right)+\theta\kappa_1A_1\rho-\theta A_1\right\}x_t. \\
&\quad+\left\{\ \tfrac{1}{2}\left(\theta-\tfrac{\theta}{\psi}\right)^2\phi_c^2+\tfrac{1}{2}\left(\theta\kappa_1A_1\phi_x\right)^2+\left(\theta\kappa_1A_2\nu-\theta A_2\right)\ \right\}\sigma_t^2
\end{aligned}$$

Thus, the Euler equation $\mathbb{E}_t M_{t\to t+1}R_{a,t\to t+1}$=1 implies

$$\begin{aligned}
\left\{\ \theta\log\delta+\left(\theta-\tfrac{\theta}{\psi}\right)\mu_c+\theta\kappa_0+\left(\kappa_1-1\right)\theta A_0+\theta\kappa_1A_2\overline{\sigma}^2\left(1-\nu\right)+\tfrac{1}{2}\left(\theta\kappa_1A_2\phi_\sigma\right)^2\ \right\} &= 0 \\
\theta-\frac{\theta}{\psi}+\theta\kappa_1A_1\rho-\theta A_1 &= 0 \\
\left\{\ \theta\kappa_1A_2\nu-\theta A_2+\tfrac{1}{2}\left(\theta-\tfrac{\theta}{\psi}\right)^2\phi_c^2+\tfrac{1}{2}\left(\theta\kappa_1A_1\phi_x\right)^2\ \right\} &= 0
\end{aligned}$$

which implies

$$\begin{aligned}
A_0 &= \frac{\left\{\ \log\delta+\left(1-\tfrac{1}{\psi}\right)\mu_c+\kappa_0+\kappa_1A_2\overline{\sigma}^2\left(1-\nu\right)+\tfrac{1}{2}\theta\left(\kappa_1A_2\phi_\sigma\right)^2\ \right\}}{\left(1-\kappa_1\right)}, \\
A_1 &= \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho}, \\
A_2 &= \frac{1}{2}\frac{\theta\left(1-\frac{1}{\psi}\right)^2\phi_c^2+\theta\left(\kappa_1A_1\phi_x\right)^2}{1-\kappa_1\rho}.
\end{aligned}$$

Next, note that

$$\log M_{t\to t+1}+\log R_{M,t\to t+1}=\theta\log\delta-\frac{\theta}{\psi}\Delta c_{t+1}-\left(1-\theta\right)\left(\kappa_0+\kappa_1z_{t+1}-z_t+\Delta c_{t+1}\right)+\kappa_{0,m}+\kappa_{1,m}z_{m,t+1}-z_{m,t}+\Delta d_{t+1}$$

which simplifies to

$$\begin{aligned}
&\log M_{t\to t+1}+\log R_{M,t\to t+1} \\
&= \left\{\begin{array}{c}\theta\log\delta-\left(1-\theta\right)\kappa_0-\left(1-\theta\right)\kappa_1A_0+\left(-\frac{\theta}{\psi}-\left(1-\theta\right)\right)\mu_c+\left(1-\theta\right)A_0+\kappa_{0,m}+\kappa_{1,m}A_{0,m}-A_{0,m}+\mu_d \\ +\left\{-\left(1-\theta\right)\kappa_1A_2+\kappa_{1,m}A_{2,m}\right\}\left\{\overline{\sigma}^2\left(1-\nu\right)\right\}\end{array}\right\} \\
&\quad+\left\{\ -\tfrac{\theta}{\psi}-\left(1-\theta\right)+\left(1-\theta\right)A_1+\phi-A_{1,m}+\left(\kappa_{1,m}A_{1,m}-\left(1-\theta\right)\kappa_1A_1\right)\rho\ \right\}x_t \\
&\quad+\left\{\left(-\frac{\theta}{\psi}-\left(1-\theta\right)\right)\phi_c\sigma_t+\phi_{d,c}\sigma_t\right\}\eta_{c,t+1}+\left\{\ \left(-\left(1-\theta\right)\kappa_1A_2+\kappa_{1,m}A_{2,m}\right)\nu+\left(1-\theta\right)A_2-A_{2,m}\ \right\}\sigma_t^2 \\
&\quad+\left\{\kappa_{1,m}A_{1,m}-\left(1-\theta\right)\kappa_1A_1\right\}\phi_x\sigma_t\eta_{x,t+1}+\left\{-\left(1-\theta\right)\kappa_1A_2+\kappa_{1,m}A_{2,m}\right\}\phi_\sigma\omega_{t+1}+\phi_d\sigma_t\eta_{d,t+1}.
\end{aligned}$$

Thus the quantity below

$$\left\{\begin{array}{c}\theta\log\delta-(1-\theta)\,\kappa_0-(1-\theta)\,\kappa_1A_0+\left(-\frac{\theta}{\psi}-(1-\theta)\right)\mu_c+(1-\theta)\,A_0+\kappa_{0,m}+\kappa_{1,m}A_{0,m}-A_{0,m}+\mu_d\\ +\left\{-(1-\theta)\,\kappa_1A_2+\kappa_{1,m}A_{2,m}\right\}\left\{\overline{\sigma}^2\,(1-\nu)\right\}+\frac{1}{2}\left(-(1-\theta)\,\kappa_1A_2+\kappa_{1,m}A_{2,m}\right)^2\phi_\sigma^2\end{array}\right\}$$
$$+\left\{\begin{array}{c}-\frac{\theta}{\psi}-(1-\theta)+(1-\theta)\,A_1-A_{1,m}+\left\{-(1-\theta)\,\kappa_1A_1+\kappa_{1,m}A_{1,m}\right\}\rho+\phi\end{array}\right\}x_t$$
$$+\left\{\begin{array}{c}\frac{1}{2}\left(-\frac{\theta}{\psi}\phi_c+\phi_{d,c}-(1-\theta)\,\phi_c\right)^2+\frac{1}{2}\left(\kappa_{1,m}A_{1,m}-(1-\theta)\,\kappa_1A_1\right)^2\phi_x^2\\ +(1-\theta)\,A_2-A_{2,m}+\left(\kappa_{1,m}A_{2,m}-(1-\theta)\,\kappa_1A_2\right)\nu+\frac{1}{2}\phi_d^2\end{array}\right\}\sigma_t^2.$$

is equal to zero. This implies that

$$\begin{aligned}A_{0,m}&=\frac{1}{(1-\kappa_{1,m})}\left\{\begin{array}{c}\mu_d+\theta\log\delta-(1-\theta)\,\kappa_0+\kappa_{0,m}+(1-\theta)\,A_0\,(1-\kappa_1)+\left(-\frac{\theta}{\psi}-(1-\theta)\right)\mu_c\\ +\left\{-(1-\theta)\,\kappa_1A_2+\kappa_{1,m}A_{2,m}\right\}\overline{\sigma}^2\,(1-\nu)+\frac{1}{2}\left(-(1-\theta)\,\kappa_1A_2+\kappa_{1,m}A_{2,m}\right)^2\phi_\sigma^2\end{array}\right\}\\ A_{1,m}&=\frac{\phi-\frac{1}{\psi}}{1-\kappa_{1,m}\rho}\\ A_{2,m}&=\frac{1}{(1-\kappa_{1,m}\nu)}\left\{\frac{1}{2}\left(\phi_{d,c}-\frac{\theta}{\psi}\phi_c-(1-\theta)\,\phi_c\right)^2+\frac{1}{2}\left(\kappa_{1,m}A_{1,m}-(1-\theta)\,\kappa_1A_1\right)^2\phi_x^2+(1-\theta)\,A_2\,(1-\kappa_1\nu)+\frac{1}{2}\phi_d^2\right\}.\end{aligned}$$

Now, observe that:

$$\begin{aligned}\log R_{M,t\to t+1}&=\kappa_{0,m}+\kappa_{1,m}z_{m,t+1}-z_{m,t}+\Delta d_{t+1}\\ &=\mathbb{A}_0^{\mathrm{er}}+\mathbb{A}_1^{\mathrm{er}}x_t+\mathbb{A}_2^{\mathrm{er}}\sigma_t^2+\kappa_{1,m}A_{1,m}\phi_x\sigma_t\eta_{x,t+1}+\kappa_{1,m}A_{2,m}\phi_\sigma\omega_{t+1}+\phi_d\sigma_t\eta_{d,t+1}+\phi_{d,c}\sigma_t\eta_{c,t+1}\end{aligned}$$

with

$$\begin{aligned}\mathbb{A}_0^{\mathrm{er}}&=\kappa_{0,m}+(\kappa_{1,m}-1)\,A_{0,m}+\mu_d+\kappa_{1,m}A_{2,m}\overline{\sigma}^2\,(1-\nu)\,,\\ \mathbb{A}_1^{\mathrm{er}}&=(\kappa_{1,m}\rho-1)\,A_{1,m}+\phi,\\ \mathbb{A}_2^{\mathrm{er}}&=(\kappa_{1,m}\nu-1)\,A_{2,m}.\end{aligned}$$

Therefore,

$$\mathbb{E}_t\left[\log R_{M,t\to t+1}\right]=\mathbb{A}_0^{\mathrm{er}}+\mathbb{A}_1^{\mathrm{er}}x_t+\mathbb{A}_2^{\mathrm{er}}\sigma_t^2$$

and

$$\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]=\mathbb{A}_0^{\mathrm{vr}}+\mathbb{A}_1^{\mathrm{vr}}\sigma_t^2$$

with

$$\begin{aligned}\mathbb{A}_0^{\mathrm{vr}}&=\kappa_{1,m}^2A_{2,m}^2\phi_\sigma^2\\ \mathbb{A}_1^{\mathrm{vr}}&=\kappa_{1,m}^2A_{1,m}^2\phi_x^2+\phi_d^2+\phi_{d,c}^2.\end{aligned}$$

Thus,

$$\mathbb{E}_t\left[R_{M,t\to t+1}^n\right]=\exp\left(n\left(\mathbb{A}_0^{\mathrm{er}}+\mathbb{A}_1^{\mathrm{er}}x_t+\mathbb{A}_2^{\mathrm{er}}\sigma_t^2\right)+\frac{n^2}{2}\left(\mathbb{A}_0^{\mathrm{vr}}+\mathbb{A}_1^{\mathrm{vr}}\sigma_t^2\right)\right)$$

Now, let us find the risk free rate.

$$\begin{aligned}\log M_{t\to t+1} &= \theta\log\delta-\frac{\theta}{\psi}\Delta c_{t+1}-(1-\theta)\log R_{a,t\to t+1}\\ &= \theta\log\delta+\left\{-\frac{\theta}{\psi}-(1-\theta)\right\}\mu_c-(1-\theta)\kappa_0-(1-\theta)\kappa_1A_0-(1-\theta)\kappa_1A_2\overline{\sigma}^2(1-\nu)+(1-\theta)A_0\\ &\quad+\left\{-\frac{\theta}{\psi}-(1-\theta)-(1-\theta)\kappa_1A_1\rho+(1-\theta)A_1\right\}x_t+(1-\kappa_1\nu)(1-\theta)A_2\sigma_t^2\\ &\quad+\left\{-\frac{\theta}{\psi}-(1-\theta)\right\}\phi_c\sigma_t\eta_{c,t+1}-(1-\theta)\kappa_1A_1\phi_x\sigma_t\eta_{x,t+1}-(1-\theta)\kappa_1A_2\phi_\sigma\omega_{t+1}.\end{aligned}$$

Thus

$$\begin{aligned}-\log R_{f,t\to t+1} &= \theta\log\delta+\left\{-\frac{\theta}{\psi}-(1-\theta)\right\}\mu_c-(1-\theta)\kappa_0-(1-\theta)\kappa_1A_0-(1-\theta)\kappa_1A_2\overline{\sigma}^2(1-\nu)+(1-\theta)A_0\\ &\quad+\left\{-\frac{\theta}{\psi}-(1-\theta)-(1-\theta)\kappa_1A_1\rho+(1-\theta)A_1\right\}x_t+\frac{1}{2}(1-\theta)^2\kappa_1^2A_2^2\phi_\sigma^2\\ &\quad+\left\{(1-\kappa_1\nu)(1-\theta)A_2\sigma_t^2+\tfrac{1}{2}\left(-\tfrac{\theta}{\psi}-(1-\theta)\right)^2\phi_c^2\sigma_t^2+\tfrac{1}{2}(1-\theta)^2\kappa_1^2A_1^2\phi_x^2\sigma_t^2\right\}.\end{aligned}$$

This simplifies to

$$\log R_{f,t\to t+1}=\mathbb{A}_0^{\mathrm{rf}}+\mathbb{A}_1^{\mathrm{rf}}x_t+\mathbb{A}_2^{\mathrm{rf}}\sigma_t^2$$

with

$$\begin{aligned}\mathbb{A}_0^{\mathrm{rf}} &= -\left\{\begin{array}{c}\theta\log\delta+\left\{-\frac{\theta}{\psi}-(1-\theta)\right\}\mu_c-(1-\theta)\kappa_0-(1-\theta)\kappa_1A_0\\ -(1-\theta)\kappa_1A_2\overline{\sigma}^2(1-\nu)+(1-\theta)A_0+\frac{1}{2}(1-\theta)^2\kappa_1^2A_2^2\phi_\sigma^2\end{array}\right\},\\ \mathbb{A}_1^{\mathrm{rf}} &= -\left\{-\frac{\theta}{\psi}-(1-\theta)-(1-\theta)\kappa_1A_1\rho+(1-\theta)A_1\right\},\\ \mathbb{A}_2^{\mathrm{rf}} &= -\left\{(1-\kappa_1\nu)(1-\theta)A_2+\frac{1}{2}\left(-\frac{\theta}{\psi}-(1-\theta)\right)^2\phi_c^2+\frac{1}{2}(1-\theta)^2\kappa_1^2A_1^2\phi_x^2\right\}.\end{aligned}$$

■

We now provide a proof of Result IA.1.

Proof. Proof of Result IA.1. From Equation IA.48, it follows that

$$\log\left(\mathbb{E}_t^*\left(R_{M,t\to t+1}^2\right)\right)=2\log R_{f,t\to t+1}+\mathbb{VAR}_t\log R_{M,t\to t+1}$$

which simplifies to

$$\log\left(\mathbb{E}_t^*\left(R^2_{M,t\to t+1}\right)\right) = \left(2\mathbb{A}_0^{\mathrm{rf}} + \mathbb{A}_0^{\mathrm{vr}}\right) + 2\mathbb{A}_1^{\mathrm{rf}} x_t + \left(2\mathbb{A}_2^{\mathrm{rf}} + \mathbb{A}_1^{\mathrm{vr}}\right)\sigma_t^2$$

since

$$\log R_{f,t\to t+1} = \mathbb{A}_0^{\mathrm{rf}} + \mathbb{A}_1^{\mathrm{rf}} x_t + \mathbb{A}_2^{\mathrm{rf}}\sigma_t^2$$

and

$$\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right] = \mathbb{A}_0^{\mathrm{vr}} + \mathbb{A}_1^{\mathrm{vr}}\sigma_t^2.$$

Thus,

$$\mathbb{VAR}_t^*\left[R_{M,t\to t+1}\right] = \exp\left(\mathbb{A}_0^{sq} + \mathbb{A}_1^{sq} x_t + \mathbb{A}_2^{sq}\sigma_t^2\right) - \exp\left(2\mathbb{A}_0^{\mathrm{rf}} + 2\mathbb{A}_1^{\mathrm{rf}} x_t + 2\mathbb{A}_2^{\mathrm{rf}}\sigma_t^2\right). \tag{IA.65}$$

with

$$\mathbb{A}_0^{sq} = 2\mathbb{A}_0^{\mathrm{rf}} + \mathbb{A}_0^{\mathrm{vr}},\ \mathbb{A}_1^{sq} = 2\mathbb{A}_1^{\mathrm{rf}}.\ \mathbb{A}_2^{sq} = 2\mathbb{A}_2^{\mathrm{rf}} + \mathbb{A}_1^{\mathrm{vr}}.$$

Next, from Equation IA.57, the price-dividend ratio is of the form

$$\log\frac{P_t}{D_t} = A_{0,m} + A_{1,m}x_t + A_{2,m}\sigma_t^2. \tag{IA.66}$$

Combining Equations IA.65 and IA.66 ends the proof.

■

Next, denote

$$\begin{aligned}
\mathbb{A}_{0,n} &= \left\{\begin{array}{c} \log R_{f,t\to t+1} + \theta\log\delta - \gamma\mu_c - (1-\theta)\kappa_0 + n\kappa_{0,m} + n\mu_d + (\kappa_{1,m}-1)\,nA_{0,m} \\ + (1-\kappa_1)(1-\theta)A_0 + (n\kappa_{1,m}A_{2,m} - (1-\theta)\kappa_1 A_2)\,\overline{\sigma}^2(1-\nu) \end{array}\right\} \\
\mathbb{A}_{1,n} &= \{-\gamma + n\phi + (1-\theta)A_1(1-\kappa_1\rho) + (\kappa_{1,m}\rho - 1)\,nA_{1,m}\} \\
\mathbb{A}_{2,n} &= \{(1-\kappa_1\nu)(1-\theta)A_2 + n(\kappa_{1,m}\nu - 1)A_{2,m}\} \\
\mathbb{A}_{3,n} &= (n\kappa_{1,m}A_{1,m}\phi_x - (1-\theta)\kappa_1 A_1\phi_x) \\
\mathbb{A}_{4,n} &= (n\phi_{d,c} - \gamma\phi_c) \\
\mathbb{A}_{5,n} &= (n\kappa_{1,m}A_{2,m}\phi_\sigma - (1-\theta)\kappa_1 A_2\phi_\sigma) \\
\mathbb{A}_{6,n} &= n\phi_d
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{A}_0^{\mathrm{er}} &= \kappa_{0,m} + (\kappa_{1,m}-1)A_{0,m} + \mu_d + \kappa_{1,m}A_{2,m}\overline{\sigma}^2(1-\nu), \\
\mathbb{A}_1^{\mathrm{er}} &= (\kappa_{1,m}\rho - 1)A_{1,m} + \phi, \\
\mathbb{A}_2^{\mathrm{er}} &= (\kappa_{1,m}\nu - 1)A_{2,m}, \\
\mathbb{A}_0^{\mathrm{vr}} &= \kappa_1^2 A_{2,m}^2\phi_\sigma^2, \\
\mathbb{A}_1^{\mathrm{vr}} &= \kappa_1^2 A_{1,m}^2\phi_x^2 + \phi_d^2 + \phi_{d,c}^2.
\end{aligned}$$

We now provide a proof for Result IA.2.

Proof. Proof of Result IA.2. Observe that

$$\mathbb{E}_t\left[R^n_{M,t\to t+1}\right] = \exp\left(n\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] + \frac{n^2}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right)$$

with

$$\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] = \mathbb{A}^{\text{er}}_0 + \mathbb{A}^{\text{er}}_1 x_t + \mathbb{A}^{\text{er}}_2\sigma^2_t \text{ and } \mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right] = \mathbb{A}^{\text{vr}}_0 + \mathbb{A}^{\text{vr}}_1\sigma^2_t. \tag{IA.67}$$

Now we will provide the formula for

$$\begin{aligned}
\mathbb{E}_t\left[R^n_{M,t\to t+1}\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] &= \mathbb{E}_t\left[\exp\left(n\log R_{M,t\to t+1}\right)\mathbb{I}_{\log R_{M,t\to t+1}>\log a}\right] \\
&= \mathbb{E}_t\left[\exp\left(n\log R_{M,t\to t+1}\right) | \log R_{M,t\to t+1} > \log a\right]\mathbb{P}_t\left[\log R_{M,t\to t+1} > \log a\right]
\end{aligned}$$

We then exploit Lemma IA.1 and show

$$\begin{aligned}
\mathbb{E}_t\left[\exp\left(n\log R_{M,t\to t+1}\right) | \log R_{M,t\to t+1} > \log a\right] &= \mathbb{E}_t\left[\exp\left(n\log R_{M,t\to t+1}\right) | n\log R_{M,t\to t+1} > n\log a\right] \\
&= \frac{\exp\left[n\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] + \frac{n^2}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right]\mathcal{N}\left[\overline{d}_{1,n}\right]}{\mathcal{N}\left[\overline{d}_{2,n}\right]}
\end{aligned}$$

where $\mathcal{N}\left[\overline{d}_{2,n}\right] = \mathbb{P}_t\left[n\log R_{M,t\to t+1} > n\log a\right]$ and

$$\begin{aligned}
\overline{d}_{1,n} &= \frac{n^2\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right] + n\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] - n\log a}{n\sqrt{\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]}} \\
\overline{d}_{2,n} &= \overline{d}_{1,n} - n\sqrt{\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]}
\end{aligned}$$

and $\mathcal{N}$ represents the CDF function for the standard normal distribution. Thus,

$$\mathbb{E}_t\left[R^n_{M,t\to t+1}\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] = \left\{\frac{\exp\left[n\left(\mathbb{A}^{\text{er}}_0 + \mathbb{A}^{\text{er}}_1 x_t + \mathbb{A}^{\text{er}}_2\sigma^2_t\right) + \frac{n^2}{2}\left(\mathbb{A}^{\text{vr}}_0 + \mathbb{A}^{\text{vr}}_1\sigma^2_t\right)\right] N\left[\overline{d}_{1,n}\right]}{\mathcal{N}\left[\overline{d}_{2,n}\right]}\right\}\mathcal{N}\left[\overline{d}_{2,n}\right]$$

where

$$\begin{aligned}
\overline{d}_{1,n} &= \frac{n^2\left(\mathbb{A}^{\text{vr}}_0 + \mathbb{A}^{\text{vr}}_1\sigma^2_t\right) + n\left(\mathbb{A}^{\text{er}}_0 + \mathbb{A}^{\text{er}}_1 x_t + \mathbb{A}^{\text{er}}_2\sigma^2_t\right) - n\log a}{n\sqrt{\mathbb{A}^{\text{vr}}_0 + \mathbb{A}^{\text{vr}}_1\sigma^2_t}} \\
\overline{d}_{2,n} &= \overline{d}_{1,n} - n\sqrt{\mathbb{A}^{\text{vr}}_0 + \mathbb{A}^{\text{vr}}_1\sigma^2_t}.
\end{aligned}$$

■

We now provide a proof of Result IA.3.

Proof. Proof of Result IA.3. Observe that

$$\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\right] = \mathbb{E}_t\left[\frac{M_{t\to t+1}}{\mathbb{E}_t\left[M_{t\to t+1}\right]}R^n_{M,t\to t+1}\right].$$

We then show:

$$\log\left(\frac{M_{t\to t+1}}{\mathbb{E}_t\left[M_{t\to t+1}\right]}R^n_{M,t\to t+1}\right) = \log R_{f,t\to t+1}+\theta\log\delta+\left\{-\frac{\theta}{\psi}-(1-\theta)\right\}\Delta c_{t+1}-(1-\theta)\,\kappa_0$$
$$-(1-\theta)\,\kappa_1 z_{t+1}+(1-\theta)\,z_t+n\kappa_{0,m}+n\kappa_{1,m}z_{m,t+1}-nz_{m,t}+n\Delta d_{t+1}$$

and

$$\log\left(\frac{M_{t\to t+1}}{\mathbb{E}_t\left[M_{t\to t+1}\right]}R^n_{M,t\to t+1}\right) = \mathbb{A}_{0,n}+\mathbb{A}_{1,n}x_t+\mathbb{A}_{2,n}\sigma_t^2+\mathbb{A}_{3,n}\sigma_t\eta_{x,t+1}+\mathbb{A}_{4,n}\sigma_t\eta_{c,t+1}+\mathbb{A}_{5,n}\omega_{t+1}+\mathbb{A}_{6,n}\sigma_t\eta_{d,t+1}$$

with

$$\begin{aligned}
\mathbb{A}_{0,n} &= \left\{\begin{array}{c}\log R_{f,t\to t+1}+\theta\log\delta-\gamma\mu_c-(1-\theta)\,\kappa_0+n\kappa_{0,m}+n\mu_d+n\kappa_{1,m}A_{0,m}-nA_{0,m}\\ -(1-\theta)\,\kappa_1A_0+(1-\theta)\,A_0-(1-\theta)\,\kappa_1A_2\overline{\sigma}^2\,(1-\nu)+n\kappa_{1,m}A_{2,m}\overline{\sigma}^2\,(1-\nu)\end{array}\right\}\\
\mathbb{A}_{1,n} &= \left\{-\gamma+n\phi-nA_{1,m}+(1-\theta)\,A_1-(1-\theta)\,\kappa_1A_1\rho+n\kappa_{1,m}A_{1,m}\rho\right\}\\
\mathbb{A}_{2,n} &= \left\{-(1-\theta)\,\kappa_1A_2\nu+(1-\theta)\,A_2-nA_{2,m}+n\kappa_{1,m}A_{2,m}\nu\right\}\\
\mathbb{A}_{3,n} &= n\kappa_{1,m}A_{1,m}\phi_x-(1-\theta)\,\kappa_1A_1\phi_x\\
\mathbb{A}_{4,n} &= n\phi_{d,c}-\gamma\phi_c\\
\mathbb{A}_{5,n} &= -(1-\theta)\,\kappa_1A_2\phi_\sigma+n\kappa_{1,m}A_{2,m}\phi_\sigma\\
\mathbb{A}_{6,n} &= n\phi_d.
\end{aligned}$$

Thus

$$\log\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\right]=\log\left(\frac{M_{t\to t+1}}{\mathbb{E}_t\left[M_{t\to t+1}\right]}R^n_{M,t\to t+1}\right)=\mu^{x,\sigma}_{n,t}+\mathbb{A}^{'}_{n,t}\eta_{x,c,\omega,d}$$

where

$$\begin{aligned}
\mu^{x,\sigma}_{n,t} &= \mathbb{A}_{0,n}+\mathbb{A}_{1,n}x_t+\mathbb{A}_{2,n}\sigma_t^2\\
\mathbb{A}^{'}_{n,t} &= \left[\mathbb{A}_{3,n}\sigma_t,\mathbb{A}_{4,n}\sigma_t,\mathbb{A}_{5,n},\mathbb{A}_{6,n}\sigma_t\right]\\
\eta^{'}_{x,c,\omega,d} &= \left[\eta_{x,t+1},\eta_{c,t+1},\omega_{t+1},\eta_{d,t+1}\right].
\end{aligned}$$

Further, from Equation IA.48, we can also infer that

$$\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\right]=\exp\left(n\log R_{f,t\to t+1}+\frac{n\,(n-1)}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right).$$

Next, recall that

$$\log R_{M,t\to t+1} = \mathbb{A}^{\mathrm{er}}_0+\mathbb{A}^{\mathrm{er}}_1x_t+\mathbb{A}^{\mathrm{er}}_2\sigma_t^2+\kappa_{1,m}A_{1,m}\phi_x\sigma_t\eta_{x,t+1}+\kappa_{1,m}A_{2,m}\phi_\sigma\omega_{t+1}+\phi_d\sigma_t\eta_{d,t+1}+\phi_{d,c}\sigma_t\eta_{c,t+1}$$

which simplifies to $n\log R_{M,t\to t+1}=\mu^{\mathrm{R}}_{n,t}+\mathbb{A}^{\mathrm{R}'}_{n,t}\eta_{x,c,\omega,d}$ with

$$\begin{aligned}\mu^{\mathrm{R}}_{n,t} &= n\mathbb{A}^{\mathrm{er}}_{0}+n\mathbb{A}^{\mathrm{er}}_{1}x_t+n\mathbb{A}^{\mathrm{er}}_{2}\sigma^2_t\\ \mathbb{A}^{\mathrm{R}'}_{n,t} &= \left[n\kappa_{1,m}A_{1,m}\phi_x\sigma_t, n\phi_{d,c}\sigma_t, n\kappa_{1,m}A_{2,m}\phi_\sigma, n\phi_d\sigma_t\right]\end{aligned}$$

Then

$$\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\right]=\exp\left(n\log R_{f,t\to t+1}+\frac{n\,(n-1)}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right).$$

Now let us compute

$$\begin{aligned}\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] &= \mathbb{E}_t\left[\frac{M_{t\to t+1}}{\mathbb{E}_t\left[M_{t\to t+1}\right]}R^n_{M,t\to t+1}\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right]\\ &= \exp\left(\mu^{x,\sigma}_{n,t}\right)\mathbb{E}_t\left[\left(\exp\left(\mathbb{A}^{'}_{n,t}\eta_{x,c,\omega,d}\right)\right)\mathbb{I}_{\left\{\mu^{\mathrm{R}}_{n,t}+\mathbb{A}^{\mathrm{R}'}_{n,t}\eta_{x,c,\omega,d}>n\log a\right\}}\right]\\ &= \exp\left(\mu^{x,\sigma}_{n,t}\right)\mathbb{E}_t\left[\left(\exp\left(\mathbb{A}^{'}_{n,t}\eta_{x,c,\omega,d}\right)\right)\mathbb{I}_{\left\{\mathbb{A}^{\mathrm{R}'}_{n,t}\eta_{x,c,\omega,d}>(n\log a)-\mu^{\mathrm{R}}_{n,t}\right\}}\right].\end{aligned}$$

Now observe that

$$\begin{aligned}\mathbb{I}_{\left\{\mathbb{A}^{\mathrm{R}'}_{n}\eta_{x,c,\omega,d}>(\log a)-\mu^{\mathrm{R}}_{n,t}\right\}} &= \mathbb{I}_{\left\{\mathbb{A}^{'}_{n,t}\mathbb{A}^{\mathrm{R}}_{n}\left(\mathbb{A}^{\mathrm{R}'}_{n}\mathbb{A}^{\mathrm{R}}_{n}\right)^{-1}\mathbb{A}^{\mathrm{R}'}_{n}\eta_{x,c,\omega,d}>\mathbb{A}^{'}_{n,t}\mathbb{A}^{\mathrm{R}}_{n}\left(\mathbb{A}^{\mathrm{R}'}_{n}\mathbb{A}^{\mathrm{R}}_{n}\right)^{-1}\left((n\log a)-\mu^{\mathrm{R}}_{n,t}\right)\right\}}\\ &= \mathbb{I}_{\left\{\mathbb{A}^{'}_{n,t}\eta_{x,c,\omega,d}>\log a^*_n\right\}}.\end{aligned}$$

with $\log a^*_n=\mathbb{A}^{'}_{n,t}\mathbb{A}^{\mathrm{R}}_{n,t}\left(\mathbb{A}^{\mathrm{R}'}_{n,t}\mathbb{A}^{\mathrm{R}}_{n,t}\right)^{-1}\left(n\log a-\mu^{\mathrm{R}}_{n,t}\right)$. We then exploit Lemma IA.1 to show

$$\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right]=\exp\left(\mu^{x,\sigma}_{n,t}\right)\mathbb{E}_t\left[\exp\left(\mathbb{A}^{'}_{n,t}\eta_{x,c,\omega,d}\right)\mathbb{I}_{\left\{\exp\left(\mathbb{A}^{'}_{n,t}\eta_{x,c,\omega,d}\right)>a^*_n\right\}}\right].$$

Thus

$$\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right]=\exp\left(\mu^{x,\sigma}_{n,t}\right)\mathcal{N}\left[\overline{d}^*_{1,n}\right]\exp\left(\frac{1}{2}\mathbb{A}^{'}_{n,t}\mathbb{A}_{n,t}\right)$$

where $\overline{d}^*_{1,n}=\frac{\mathbb{A}^{'}_{n,t}\mathbb{A}_{n,t}-\log a^*_n}{\sqrt{\mathbb{A}^{'}_{n,t}\mathbb{A}_{n,t}}}$ and $\overline{d}^*_{2,n}=\overline{d}^*_{1,n}-\sqrt{\mathbb{A}^{'}_{n,t}\mathbb{A}_{n,t}}$ and $\mathbb{E}^*_t\left[\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right]=N\left[\overline{d}^*_{2,n}\right]$.

■

IA.7.3.2 Bollerslev, Tauchen, and Zhou (2009)

Bollerslev, Tauchen, and Zhou (2009) use the Epstein and Zin (1989) SDF and approximate $R_{a,t\to t+1}$ in Equation IA.49 by the market return$R_{M,t\to t+1}$. They set up an economy governed by the following time series

$$\begin{aligned}g_{t+1} &= d_{t+1}=\mu_g+\sigma_{g,t}z_{g,t+1},\\ \sigma^2_{g,t+1} &= a_\sigma+\rho_\sigma\sigma^2_{g,t}+\sqrt{q_t}z_{\sigma,t+1},\\ q_{t+1} &= a_q+\rho_q q_t+\phi_q\sqrt{q_t}z_{q,t+1}\end{aligned}\tag{IA.68}$$

where g_{t+1} represents consumption growth. There are two state variables in their framework, $\sigma^2_{g,t}$ and q_t.

Bollerslev, Tauchen, and Zhou (2009) build a model that combines stochastic volatility with Epstein and Zin (1989) preferences to provide a theoretical foundation for the empirical fact that the variance risk premium can be used to forecast market returns at short horizons. Their model includes two state variables ($\sigma^2_{g,t}$ and q_t), so we use $\log(P_t/E_t)$ and $\mathbb{M}^{*(2)}_{t\to T}$ to extract implied state variables at each date using Result IA.4.

Summary statistics for the extracted state variables can be found in Table IA.5. Average state variable values are similar in magnitude to and fall within the confidence intervals implied by the calibrated model. Data-implied $\sigma^2_{g,t}$ and q_t standard deviations are higher than those implied by the models and fall outside the model-implied confidence intervals. The data-implied $\sigma^2_{g,t}$ (g_t) autocorrelation is lower (higher) than the model-implied value.

Given values for the data-implied state variables, we can compute the physical and risk-neutral moments necessary for the risk premium decomposition using model-implied moments from Results IA.5 and IA.6, which can be found below. Table IA.6 provides summary statistics for the market risk premium decomposition and Table IA.7 provides the average difference between the data-implied risk premia and those implied by the Bollerslev, Tauchen, and Zhou (2009) model. The risk premia implied by Bollerslev, Tauchen, and Zhou (2009) are quite surprising. For instance, the total risk premium is often negative. We confirm that this also happens in the simulated model in approximately 44% of the simulated values.[44] We also find that the data-implied risk premium standard deviation is quite high at about 145% (annualized), which is higher than the model-implied value of 19% and is likely caused by the relatively high data-implied state variable volatility compared to the model-implied values. Despite the high fraction of negative values, the unconditional risk premium implied by the simulated model is more reasonable at 8.67%. This is slightly lower than the average risk premium implied using extracted state variables (10.71%),[45] although the discrepancy is small relative to the risk premium volatility based on our extracted state variables. Unconditionally, the central risk premium contributes about 80% of the total risk premium, although the contribution varies significantly over time. For instance, during the 2008 Financial Crisis, the upside risk premium actually contributed up to almost 90% of the total risk premium. During the Dot-com bust, the total risk premium implied by the model actually becomes negative, and the downside risk premium actually comprised approximately 90% of this premium. That is, the model implies that during this period investors were willing to accept

[44] This finding is consistent with related issues documented by Bekaert, Engstrom, and Ermolov (2020) with respect to the Bollerslev, Tauchen, and Zhou (2009) model.

[45] This discrepancy is primarily caused by the fact that the extracted state variables are more volatile than the model-implied state variables. Since the risk premium is related to exponentials of functions that are affine in state variables, Jensen's inequality effects cause the risk premium based on our extracted state variables to be higher than that implied by the simulated model using the state variable processes from the original calibrated model.

negative expected returns for exposure to downside risk.

Main Results Given this setup, we show the following results.

Result IA.4. *Given the state variables $\sigma_{g,t}^2$ and q_t, the Bollerslev, Tauchen, and Zhou (2009) model-implied log price-dividend ratio is given by*

$$\log \frac{P_t}{D_t} = A_0 + A_\sigma \sigma_{g,t}^2 + A_q q_t, \tag{IA.69}$$

and the risk-neutral market return variance is given by

$$\mathbb{M}^{*(2)}_{t\to t+1}[A] = \exp\left(\mathbb{B}_0^{sq} + \mathbb{B}_1^{sq}\sigma_{g,t}^2 + \mathbb{B}_2^{sq} q_t\right) - \exp\left(\mathbb{A}_0^{rf} + \mathbb{A}_1^{rf}\sigma_{g,t}^2 + \mathbb{A}_2^{rf} q_t\right). \tag{IA.70}$$

The coefficients A_0, A_σ, A_q, $\mathbb{B}_0^{sq}$, $\mathbb{B}_1^{sq}$, and $\mathbb{B}_2^{sq}$ are defined below.

Proof. See below. ■

Result IA.5. *The conditional non-central physical moment of the market return is*

$$\mathbb{E}_t\left[R^n_{M,t\to t+1}\right] = \exp\left(n\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] + \frac{n^2}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right).$$

where

$$\begin{aligned}\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] \quad = \quad & \kappa_0 + \kappa_1 A_0 - A_0 + \kappa_1 A_\sigma a_\sigma + \kappa_1 A_q a_q + \mu_g \\ & + (\kappa_1\rho_\sigma - 1) A_\sigma \sigma_{g,t}^2 + (\kappa_1 \rho_q - 1) A_q q_t\end{aligned}$$

and

$$\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right] = (\kappa_1 A_\sigma)^2 q_t + (\kappa_1 A_q \phi_q)^2 q_t + \sigma_{g,t}^2.$$

Using Lemma IA.1, the conditional non-central truncated physical moment of the market return is

$$\begin{aligned}\mathbb{E}_t\left[R^n_{M,t\to t+1}\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] \quad & = \quad \mathbb{E}_t\left[e^{n\log R_{M,t\to t+1}}\mathbb{I}_{\{n\log R_{M,t\to t+1}>n\log a\}}\right] \\ & = \quad \left(\exp\left\{n\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] + \frac{n^2}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right\}\right)\mathcal{N}\left[\overline{d}_1\right]\end{aligned}$$

where

$$\overline{d}_1 = \frac{n\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] + n^2\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right] - n\log a}{n\sqrt{\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]}}$$

with $\overline{d}_2 = \overline{d}_1 - n\sqrt{\mathbb{VAR}_t\left[\log R_{t+1}\right]}$. All parameters are defined below.

Proof. See below. ■

Result IA.6. *The conditional non-central risk neutral moment of the market return is*

$$\mathbb{E}_t^*\left[R_{M,t\to t+1}^n\right] = \exp\left\{n\log R_{f,t\to t+1} + \frac{n(n-1)}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right\}$$

The conditional non-central truncated risk neutral moment of the market return is

$$\mathbb{E}_t^*\left[R_{M,t\to t+1}^n \mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] = e^{A_0^{mr}}\mathcal{N}\left[\overline{d}_1\right]\exp\left\{\frac{1}{2}\lambda_t^{m\prime}\lambda_t^m\right\}$$

where

$$\overline{d}_1 = \frac{\lambda_t^{m\prime}\lambda_t^m - \lambda\left(n\log a - \mathbb{A}_{0,t}^r\right)}{\sqrt{\lambda_t^{m\prime}\lambda_t^m}}$$

and $\overline{d}_2 = \overline{d}_1 - \sqrt{\lambda_t^{m\prime}\lambda_t^m}$. *All parameters are defined below.*

Proof. See below. ∎

Derivations and Proofs

We can use the Campbell and Shiller (1988) approximation to write the log gross return as

$$\log R_{a,t\to t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}.$$

In this model, there is no distinction between the aggregate and market returns, so we denote $r_{t+1} = \log R_{M,t\to t+1} = \log R_{a,t\to t+1}$. The log price-consumption ratio is given by

$$z_t = A_0 + A_\sigma\sigma_{g,t}^2 + A_q q_t.$$

The representative agent's first-order conditions imply

$$\log\left(\mathbb{E}_t\exp\left\{m_{t+1}+r_{t+1}\right\}\right) = 0.$$

Under this model, we can write

$$\begin{aligned}
m_{t+1}+r_{t+1} &= \theta\log\delta - \frac{\theta}{\psi}g_{t+1} + \theta r_{t+1} \\
&= \theta\log\delta - \frac{\theta}{\psi}\mu_g - \frac{\theta}{\psi}\sigma_{g,t}z_{g,t+1} + \theta\kappa_0 + \theta\kappa_1 z_{t+1} - \theta z_t + \theta g_{t+1} \\
&= \theta\log\delta - \frac{\theta}{\psi}\mu_g - \frac{\theta}{\psi}\sigma_{g,t}z_{g,t+1} \\
&\quad +\theta\kappa_0 + \theta\kappa_1 A_0 + \theta\kappa_1 A_\sigma a_\sigma + \theta\kappa_1 A_\sigma\rho_\sigma\sigma_{g,t}^2 + \theta\kappa_1 A_\sigma\sqrt{q_t}z_{\sigma,t+1} \\
&\quad +\theta\kappa_1 A_q a_q + \theta\kappa_1 A_q\rho_q q_t + \theta\kappa_1 A_q\phi_q\sqrt{q_t}z_{q,t+1} \\
&\quad -\theta A_0 - \theta A_\sigma\sigma_{g,t}^2 - \theta A_q q_t + \theta\mu_g + \theta\sigma_{g,t}z_{g,t+1}
\end{aligned}$$

which simplifies to

$$
\begin{aligned}
m_{t+1}+r_{t+1} \quad = \quad & \left\{\theta\log\delta-\frac{\theta}{\psi}\mu_g+\theta\kappa_0+\theta\kappa_1A_0+\theta\kappa_1A_\sigma a_\sigma+\theta\kappa_1A_qa_q+\theta\mu_g-\theta A_0\right\} \\
& +\left\{\theta\sigma_{g,t}-\frac{\theta}{\psi}\sigma_{g,t}\right\}z_{g,t+1}+\left\{\theta\kappa_1A_\sigma\rho_\sigma-\theta A_\sigma\right\}\sigma_{g,t}^2+\theta\kappa_1A_\sigma\sqrt{q_t}z_{\sigma,t+1} \\
& +\left\{\theta\kappa_1A_q\rho_q-\theta A_q\right\}q_t+\theta\kappa_1A_q\phi_q\sqrt{q_t}z_{q,t+1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \left\{\ \theta\log\delta-\tfrac{\theta}{\psi}\mu_g+\theta\kappa_0+\theta\kappa_1A_0+\theta\kappa_1A_\sigma a_\sigma+\theta\kappa_1A_qa_q+\theta\mu_g-\theta A_0\ \right\} \\
& +\left\{\theta\kappa_1A_\sigma\rho_\sigma-\theta A_\sigma+\frac{1}{2}\left\{\theta-\frac{\theta}{\psi}\right\}^2\right\}\sigma_{g,t}^2 \\
& +\left\{\frac{1}{2}\left(\theta\kappa_1A_\sigma\right)^2+\left(\theta\kappa_1\rho_q-\theta\right)A_q+\frac{1}{2}\left(\theta\kappa_1\phi_q\right)^2A_q^2\right\}q_t \\
= \quad & 0
\end{aligned}
$$

which implies that

$$
\begin{aligned}
A_0 \quad &= \quad \frac{\theta\log\delta-\frac{\theta}{\psi}\mu_g+\theta\kappa_0+\theta\kappa_1A_\sigma a_\sigma+\theta\kappa_1A_qa_q+\theta\mu_g}{\left(1-\kappa_1\right)\theta}, \\
A_\sigma \quad &= \quad \frac{1}{2}\frac{\left(1-\gamma\right)^2}{\left(1-\kappa_1\rho_\sigma\right)\theta},
\end{aligned}
$$

and

$$
\left(\kappa_1\rho_q-1\right)A_q+\theta\left(\kappa_1A_\sigma\right)^2+\theta\left(\kappa_1\phi_q\right)^2A_q^2=0.
$$

Consequently,

$$
A_q=\frac{\left(1-\kappa_1\rho_q\right)\pm\sqrt{\left(1-\kappa_1\rho_q\right)^2-\theta^2\kappa_1^4A_\sigma^2\phi_q^2}}{\theta\left(\kappa_1\phi_q\right)^2}.
$$

Now, let us find the risk-free rate. Notice that

$$
\begin{aligned}
m_{t+1} \quad &= \quad \theta\log\delta-\frac{\theta}{\psi}g_{t+1}-\left(1-\theta\right)r_{t+1} \\
&= \quad \left\{\begin{array}{c}\theta\log\delta-\frac{\theta}{\psi}\mu_g-\left(1-\theta\right)\kappa_0-\left(1-\theta\right)\mu_g-\kappa_1\left(1-\theta\right)A_0 \\ -A_\sigma\kappa_1\left(1-\theta\right)a_\sigma-A_q\kappa_1\left(1-\theta\right)a_q+\left(1-\theta\right)A_0\end{array}\right\} \\
& \quad\ -\gamma\sigma_{g,t}z_{g,t+1}+\left\{\left(1-\theta\right)A_\sigma-A_\sigma\kappa_1\left(1-\theta\right)\rho_\sigma\right\}\sigma_{g,t}^2 \\
& \quad\ -A_\sigma\kappa_1\left(1-\theta\right)\sqrt{q_t}z_{\sigma,t+1} \\
& \quad\ +\left\{\left(1-\theta\right)A_q-A_q\kappa_1\left(1-\theta\right)\rho_q\right\}q_t-A_q\kappa_1\left(1-\theta\right)\phi_q\sqrt{q_t}z_{q,t+1}.
\end{aligned}
$$

Hence

$$\begin{aligned}\log R_{f,t\to t+1} &= -\left\{\begin{array}{c}\theta\log\delta-\frac{\theta}{\psi}\mu_g-(1-\theta)\,\kappa_0-(1-\theta)\,\mu_g\\ +(1-\theta)\,A_0\,(1-\kappa_1)-A_\sigma\kappa_1\,(1-\theta)\,a_\sigma\\ -A_q\kappa_1\,(1-\theta)\,a_q\end{array}\right\}\\ &\quad -\left\{\frac{1}{2}\gamma^2+(1-\theta)\,A_\sigma\,(1-\kappa_1\rho_\sigma)\right\}\sigma_{g,t}^2\\ &\quad -\left\{\begin{array}{c}\frac{1}{2}\left(A_\sigma\kappa_1\,(1-\theta)\right)^2+(1-\theta)\,A_q\,(1-\kappa_1\rho_q)\\ +\frac{1}{2}\left(A_q\kappa_1\,(1-\theta)\,\phi_q\right)^2\end{array}\right\}q_t.\end{aligned}$$

Hence

$$\log R_{f,t\to t+1}=\mathbb{A}_0^{\mathrm{rf}}+\mathbb{A}_1^{\mathrm{rf}}\sigma_{g,t}^2+\mathbb{A}_2^{\mathrm{rf}}q_t \tag{IA.71}$$

with

$$\begin{aligned}\mathbb{A}_0^{\mathrm{rf}} &= -\left\{\begin{array}{c}\theta\log\delta-\frac{\theta}{\psi}\mu_g-(1-\theta)\,\kappa_0-(1-\theta)\,\mu_g\\ +(1-\theta)\,A_0\,(1-\kappa_1)-A_\sigma\kappa_1\,(1-\theta)\,a_\sigma\\ -A_q\kappa_1\,(1-\theta)\,a_q\end{array}\right\},\\ \mathbb{A}_1^{\mathrm{rf}} &= -\left\{\frac{1}{2}\gamma^2+(1-\theta)\,A_\sigma\,(1-\kappa_1\rho_\sigma)\right\},\\ \mathbb{A}_2^{\mathrm{rf}} &= -\left\{\begin{array}{c}\frac{1}{2}\left(A_\sigma\kappa_1\,(1-\theta)\right)^2+(1-\theta)\,A_q\,(1-\kappa_1\rho_q)\\ +\frac{1}{2}\left(A_q\kappa_1\,(1-\theta)\,\phi_q\right)^2\end{array}\right\}.\end{aligned}$$

Next, note that

$$\begin{aligned}\log R_{M,t\to t+1} &= \kappa_0+\kappa_1 z_{t+1}-z_t+g_{t+1}\\ &= \kappa_0+\kappa_1 A_0-A_0+\kappa_1 A_\sigma a_\sigma+\kappa_1 A_q a_q+\mu_g\\ &\quad +(\kappa_1\rho_\sigma-1)\,A_\sigma\sigma_{g,t}^2+\kappa_1 A_\sigma\sqrt{q_t}z_{\sigma,t+1}\\ &\quad +(\kappa_1\rho_q-1)\,A_q q_t+\kappa_1 A_q\phi_q\sqrt{q_t}z_{q,t+1}\\ &\quad +\sigma_{g,t}z_{g,t+1}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] &= \kappa_0+\kappa_1 A_0-A_0+\kappa_1 A_\sigma a_\sigma+\kappa_1 A_q a_q+\mu_g\\ &\quad +(\kappa_1\rho_\sigma-1)\,A_\sigma\sigma_{g,t}^2+(\kappa_1\rho_q-1)\,A_q q_t\end{aligned}$$

and

$$\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right] = \left(\kappa_1 A_\sigma\right)^2 q_t + \left(\kappa_1 A_q \phi_q\right)^2 q_t + \sigma_{g,t}^2.$$

We now provide a proof of Result IA.4.

Proof. Proof of Result IA.4. From Equation IA.48, it follows that

$$\log\left(\mathbb{E}_t^*\left(R_{M,t\to t+1}^2\right)\right) = 2\log R_{f,t\to t+1} + \mathbb{VAR}_t \log R_{M,t\to t+1}$$

Thus,

$$\log\left(\mathbb{E}_t^*\left(R_{M,t\to t+1}^2\right)\right) = 2\mathbb{A}_0^{\mathrm{rf}} + 2\mathbb{A}_1^{\mathrm{rf}}\sigma_{g,t}^2 + 2\mathbb{A}_2^{\mathrm{rf}} q_t + \left(\kappa_1 A_\sigma\right)^2 q_t + \left(\kappa_1 A_q \phi_q\right)^2 q_t + \sigma_{g,t}^2.$$

Hence,

$$\log\left(\mathbb{E}_t^*\left(R_{M,t\to t+1}^2\right)\right) = 2\mathbb{A}_0^{\mathrm{rf}} + \left(2\mathbb{A}_2^{\mathrm{rf}} + \left(\kappa_1 A_\sigma\right)^2 + \left(\kappa_1 A_q \phi_q\right)^2\right) q_t + \left(1 + 2\mathbb{A}_1^{\mathrm{rf}}\right)\sigma_{g,t}^2$$

which simplifies to

$$\log\left(\mathbb{E}_t^*\left[R_{M,t\to t+1}^2\right]\right) = \mathbb{B}_0^{sq} + \mathbb{B}_1^{sq}\sigma_{g,t}^2 + \mathbb{B}_2^{sq} q_t,$$

where

$$\begin{aligned}
\mathbb{B}_0^{sq} &= 2\mathbb{A}_0^{\mathrm{rf}},\\
\mathbb{B}_1^{sq} &= 1 + 2\mathbb{A}_1^{\mathrm{rf}},\\
\mathbb{B}_2^{sq} &= 2\mathbb{A}_2^{\mathrm{rf}} + \left(\kappa_1 A_\sigma\right)^2 + \left(\kappa_1 A_q \phi_q\right)^2.
\end{aligned}$$

The variance of the market return under the risk neutral measure is

$$\mathbb{VAR}_t^*\left[R_{M,t\to t+1}\right] = \exp\left\{\mathbb{B}_0^{sq} + \mathbb{B}_1^{sq}\sigma_{g,t}^2 + \mathbb{B}_2^{sq} q_t\right\} - \exp\left\{2\mathbb{A}_0^{\mathrm{rf}} + 2\mathbb{A}_1^{\mathrm{rf}}\sigma_{g,t}^2 + 2\mathbb{A}_2^{\mathrm{rf}} q_t\right\}. \tag{IA.72}$$

Recall that the log price consumption ratio follows

$$\log\frac{P_t}{D_t} = A_0 + A_\sigma \sigma_{g,t}^2 + A_q q_t. \tag{IA.73}$$

This ends the proof.

∎

We now provide a proof of Result IA.5.

Proof. Proof of Result IA.5. Note that the log return, which is given by expression

$$\begin{aligned}
\log R_{M,t\to t+1} = \;& \kappa_0 + \kappa_1 A_0 - A_0 + \kappa_1 A_\sigma a_\sigma + \kappa_1 A_q a_q + \mu_g\\
& + \left(\kappa_1 \rho_\sigma - 1\right) A_\sigma \sigma_{g,t}^2 + \kappa_1 A_\sigma \sqrt{q_t} z_{\sigma,t+1}\\
& + \left(\kappa_1 \rho_q - 1\right) A_q q_t + \kappa_1 A_q \phi_q \sqrt{q_t} z_{q,t+1}\\
& + \sigma_{g,t} z_{g,t+1},
\end{aligned}$$

is normally distributed. Thus,

$$\mathbb{E}_t\left[R_{M,t\to t+1}^n\right] = \exp\left(n\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] + \frac{n^2}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right) \tag{IA.74}$$

with

$$\mathbb{E}_t\left[\log R_{M,t\to t+1}\right] = \kappa_0+\kappa_1 A_0 - A_0+\kappa_1 A_\sigma a_\sigma+\kappa_1 A_q a_q+\mu_g + \left(\kappa_1\rho_\sigma-1\right)A_\sigma \sigma_{g,t}^2+\left(\kappa_1\rho_q-1\right)A_q q_t$$

and

$$\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right] = \left(\kappa_1 A_\sigma\right)^2 q_t+\left(\kappa_1 A_q\phi_q\right)^2 q_t+\sigma_{g,t}^2.$$

Using Lemma IA.1, the conditional truncated physical moment of the market return is

$$\begin{aligned}\mathbb{E}_t\left[R_{M,t\to t+1}^n \mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] &= \mathbb{E}_t\left[e^{n\log R_{M,t\to t+1}}\mathbb{I}_{\{n\log R_{M,t\to t+1}>n\log a\}}\right]\\ &= \left(\exp\left\{\mathbb{E}_t\left[\log R_{M,t\to t+1}\right]+\frac{1}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]\right\}\right)\mathcal{N}\left[\overline{d}_1\right]\end{aligned}$$

where

$$\overline{d}_1=\frac{\mathbb{E}_t\left[\log R_{M,t\to t+1}\right]+\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]-n\log a}{\sqrt{\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]}}$$

with $\overline{d}_2=\overline{d}_1-\sqrt{\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right]}$.

■

We now provide a proof of Result IA.6.

Proof. Proof of Result IA.6. Recall that

$$\log\left(\mathbb{E}_t^*\left[R_{M,t\to t+1}^n\right]\right)=n\log R_{f,t\to T}+\frac{n\left(n-1\right)}{2}\mathbb{VAR}_t\left[\log R_{M,t\to t+1}\right].$$

To derive expressions for the truncated physical and risk-neutral moments, observe that

$$\mathbb{E}_t^*\left[R_{M,t\to t+1}^n\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right]=\mathbb{E}_t\left[e^{\left(m_{t+1}+n\log R_{M,t\to t+1}\right)}\mathbb{I}_{\{n\log R_{M,t\to t+1}>n\log a\}}\right].$$

Thus,

$$\begin{aligned}& m_{t+1}+n\log R_{M,t\to t+1}\\ =\; & \theta\log\delta-\frac{\theta}{\psi}g_{t+1}-\left(1-\theta\right)r_{t+1}+nr_{t+1}\\ =\; & \theta\log\delta-\frac{\theta}{\psi}g_{t+1}+\left(n+\theta-1\right)r_{t+1}\\ =\; & \left\{\begin{array}{c}\theta\log\delta+\left(n+\theta-1\right)\kappa_0+\left(\left(n+\theta-1\right)-\frac{\theta}{\psi}\right)\mu_g+\left(n+\theta-1\right)\kappa_1 A_0\\ -\left(n+\theta-1\right)A_0+\left(n+\theta-1\right)\kappa_1 A_\sigma a_\sigma+\left(n+\theta-1\right)\kappa_1 A_q a_q\end{array}\right\}\\ & +\left\{\left(n+\theta-1\right)\kappa_1 A_q\rho_q-\left(n+\theta-1\right)A_q\right\}q_t+\left\{\left(n+\theta-1\right)\kappa_1 A_\sigma\rho_\sigma-\left(n+\theta-1\right)A_\sigma\sigma_{g,t}^2\right\}\sigma_{g,t}^2\\ & +\left(n+\theta-1\right)\kappa_1 A_\sigma\sqrt{q_t}z_{\sigma,t+1}+\left(\left(n+\theta-1\right)-\frac{\theta}{\psi}\right)\sigma_{g,t}z_{g,t+1}+\left(n+\theta-1\right)\kappa_1 A_q\phi_q\sqrt{q_t}z_{q,t+1}.\end{aligned}$$

Hence

$$m_{t+1} + n \log R_{M,t\to t+1} = A_0^{mr} + \lambda_t^{m'} \mathbf{z}_t^{mr},$$

where

$$\begin{aligned}
\mathbb{A}_0^{mr} \quad &= \quad \left\{ \begin{array}{c} \theta \log \delta + (n+\theta-1)\,\kappa_0 + \left((n+\theta-1) - \frac{\theta}{\psi}\right)\mu_g + (n+\theta-1)\,\kappa_1 A_0 \\ -\,(n+\theta-1)\,A_0 + (n+\theta-1)\,\kappa_1 A_\sigma a_\sigma + (n+\theta-1)\,\kappa_1 A_q a_q \end{array} \right\} \\
&\left\{ \ (n+\theta-1)\,\kappa_1 A_q \rho_q - (n+\theta-1)\,A_q \ \right\} q_t + \left\{ \ (n+\theta-1)\,\kappa_1 A_\sigma \rho_\sigma - (n+\theta-1)\,A_\sigma \sigma_{g,t}^2 \ \right\} \sigma_{g,t}^2
\end{aligned}$$

and

$$\lambda_t^m = \begin{bmatrix} (n+\theta-1)\,\kappa_1 A_\sigma \sqrt{q_t} \\ \left((n+\theta-1) - \frac{\theta}{\psi}\right)\sigma_{g,t} \\ (n+\theta-1)\,\kappa_1 A_q \phi_q \sqrt{q_t} \end{bmatrix} \text{ and } \mathbf{z}_t^{mr} = \begin{bmatrix} z_{\sigma,t+1} \\ z_{g,t+1} \\ z_{q,t+1} \end{bmatrix}.$$

Next,

$$\log R_{M,t\to t+1} = \mathbb{A}_{0,t}^r + \lambda_t^{r'} \mathbf{z}_t^{mr}$$

where

$$\mathbb{A}_{0,t}^r \quad = \quad \kappa_0 + \kappa_1 A_0 - A_0 + \kappa_1 A_\sigma a_\sigma + \kappa_1 A_q a_q + \mu_g + (\kappa_1 \rho_\sigma - 1)\,A_\sigma \sigma_{g,t}^2 + (\kappa_1 \rho_q - 1)\,A_q q_t$$

and

$$\lambda_t^{r'} = \begin{bmatrix} \kappa_1 A_\sigma \sqrt{q_t} \\ \sigma_{g,t} \\ \kappa_1 A_q \phi_q \sqrt{q_t} \end{bmatrix}.$$

Finally

$$\begin{aligned}
\mathbb{E}_t^*\left[R_{M,t\to t+1}^n \mathbb{I}_{\left\{R_{M,t\to t+1} > a\right\}}\right] \quad &= \quad \mathbb{E}_t\left[e^{A_0^{mr} + \lambda_t^{m'} \mathbf{z}_t^{mr}} \mathbb{I}_{\left\{\mathbb{A}_{0,t}^r + \lambda_t^{r'} \mathbf{z}_t^{mr} > n \log a\right\}}\right] \\
&= \quad e^{A_0^{mr}} \mathbb{E}_t\left[e^{\lambda_t^{m'} \mathbf{z}_t^{mr}} \mathbb{I}_{\left\{\lambda_t^{m'} \mathbf{z}_t^{mr} > \lambda\left(n \log a - \mathbb{A}_{0,t}^r\right)\right\}}\right]
\end{aligned}$$

where

$$\lambda_t^{m'} = \lambda \lambda_t^{r'}.$$

Hence,

$$\lambda = \left(\lambda_t^{m'} \lambda_t^r\right)\left(\lambda_t^{r'} \lambda_t^r\right)^{-1}.$$

We then exploit Lemma IA.1 to show

$$\mathbb{E}_t\left[e^{\lambda_t^{m\prime}\mathbf{z}_t^{mr}}\mathbb{I}_{\left\{\lambda_t^{m\prime}\mathbf{z}_t^{mr}>\lambda\left(n\log a-\mathbb{A}_{0,t}^{r}\right)\right\}}\right]=\left(\exp\left\{\frac{1}{2}\lambda_t^{m\prime}\lambda_t^{m}\right\}\right)\mathcal{N}\left[\overline{d}_1\right]$$

with

$$\overline{d}_1=\frac{\lambda_t^{m\prime}\lambda_t^{m}-\lambda\left(n\log a-\mathbb{A}_{0,t}^{r}\right)}{\sqrt{\lambda_t^{m\prime}\lambda_t^{m}}}$$

and

$$\overline{d}_2=\overline{d}_1-\sqrt{\lambda_t^{m\prime}\lambda_t^{m}}.$$

■

IA.7.3.3 Drechsler and Yaron (2011)

The key variables that drive the economy in Drechsler and Yaron (2011) are summarized by

$$Y_{t+1}=\mu+FY_t+G_t z_{t+1}+J_{t+1}$$

where $Y_t^{\prime}=(\Delta c_t, x_t, \overline{\sigma}_t^2, \sigma_t^2, \Delta d_t)$, z_{t+1} is a vector of standard normal shocks, and J_{t+1} is a compound Poisson process.

Main Results

Result IA.7. *Given the vector Y_t, the Drechsler and Yaron (2011) model-implied log price-dividend ratio is*

$$\log\frac{P_t}{D_t}=A_{0,m}+A_m^{\prime}Y_t, \tag{IA.75}$$

and the non-central risk-neutral market return moments are

$$\mathbb{E}_t^*\left[R_{M,t\to t+1}^{n}\right]=R_{f,t\to t+1}\exp\left\{\begin{array}{c}\theta\log\delta-(1-\theta)\left(\kappa_0+A_0\left(\kappa_1-1\right)\right)+\mathbf{f}\left(\Lambda^*\right)\\ +\left(n\kappa_{0,m}+n\kappa_{1,m}A_{0,m}-nA_{0,m}\right)\\ +\left(-\left(\theta-1\right)A-nA_m+\mathbf{g}\left(\Lambda^*\right)\right)Y_t\end{array}\right\}$$

with

$$R_{f,t\to t+1}=\exp\left\{\begin{array}{c}-\left(\theta\log\delta+\mathbf{f}\left(-\Lambda\right)-(1-\theta)\left(\kappa_0+A_0\left(\kappa_1-1\right)\right)\right)\\ -\left(\left(1-\theta\right)A+\mathbf{g}\left(-\Lambda\right)\right)^{\prime}Y_t\end{array}\right\}$$

and

$$\Lambda = \gamma e_c + (1-\theta)\,\kappa_1 A \text{ and } \Lambda^* + \Lambda = n\kappa_{1,m}A_m + ne_d.$$

The coefficients, A_0, A, $A_{0,m}$ and A_m are given below. e_c and e_d are column vectors that select log consumption growth and log dividend growth in Y_{t+1}. $R_{f,t\to t+1}$ is the model-implied risk-free rate. Given $\mathbb{E}_t^*\left[R^k_{M,t\to t+1}\right]$, $\mathbb{M}^{*(n)}_{t\to t+1}$ can be computed using Equation 17 by setting $x = R_{M,t\to t+1}$ and $x_s = R_{f,t\to t+1}$, then taking expectations under the risk-neutral measure:

$$\mathbb{M}^{*(n)}_{t\to t+1} \quad = \quad \sum_{k=0}^{n} \frac{n!\,(-1)^{n-k}}{(n-k)!k!} \left(\mathbb{E}_t^*\left[R_{M,t\to t+1}\right]\right)^{n-k} \left(\mathbb{E}_t^*\left[R^k_{M,t\to t+1}\right]\right). \tag{IA.76}$$

Proof. See below. ∎

Result IA.8. *The non-central physical moment of the market return is*

$$\mathbb{E}_t R^n_{M,t+1} = \exp\left\{\begin{array}{c} \left(n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}\right) + \mathbf{f}\left(n\kappa_{1,m}A_m + ne_d\right) \\ \\ +\left\{\mathbf{g}\left(n\kappa_{1,m}A_m + ne_d\right) - nA_m\right\}' Y_t \end{array}\right\}$$

and the truncated moment is

$$\mathbb{E}_t\left[R^n_{M,t+1}1_{R_{M,t+1}>a}\right] = \mathbb{E}_t\left\{\left(\exp\left\{\mathcal{A}_{y,t} + \left(\Lambda^*+\Lambda\right)' Y_{t+1}\right\}\right)\mathbb{I}_{\left\{\mathcal{A}_{y,t}+\left(\Lambda^*+\Lambda\right)'Y_{t+1}>n\log a\right\}}\right\}$$

with

$$\mathcal{A}_{y,t} = \left(n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}\right) - nA_m' Y_t$$

where

$$\Lambda = \gamma e_c + (1-\theta)\,\kappa_1 A \text{ and } \Lambda^* + \Lambda = n\kappa_{1,m}A_m + ne_d.$$

All parameters are defined below.

Proof. See below. ∎

Result IA.9. *The non-central risk-neutral moment of the market return is*

$$\mathbb{E}_t^* R^n_{M,t\to t+1} = R_{f,t+1}\exp\left\{\begin{array}{c} \theta\log\delta - (1-\theta)\left[\kappa_0 + A_0\left(\kappa_1 - 1\right)\right] \\ \\ +\left(n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}\right) + \mathbf{f}\left(\Lambda^*\right) \\ \\ +\left\{-\left(\theta-1\right)A - nA_m + \mathbf{g}\left(\Lambda^*\right)\right\} Y_t \end{array}\right\}$$

and the truncated risk-neutral moment is

$$\mathbb{E}_t^*\left[R_{M,t+1}^n 1_{R_{M,t+1}>a}\right] = R_{f,t+1}\mathbb{E}_t^*\left[e^{\mathcal{A}_{y,t}^* + \Lambda^{*'} Y_{t+1}} 1_{\mathcal{A}_{y,t}+(\Lambda^*+\Lambda)' Y_{t+1} > n \log a}\right]$$

with

$$\mathcal{A}_{y,t} = (n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}) - nA_m' Y_t$$

and

$$\mathcal{A}_{y,t}^* = \left\{\begin{array}{c} \theta \log \delta - (1-\theta)\left[\kappa_0 + A_0\left(\kappa_1 - 1\right)\right] \\ + \left(n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}\right) \\ -\left(\theta - 1\right) A' Y_t - nA_m' Y_t \end{array}\right\}$$

where

$$\Lambda = \gamma e_c + (1-\theta)\,\kappa_1 A \text{ and } \Lambda^* + \Lambda = n\kappa_{1,m}A_m + ne_d.$$

All parameters are defined below.

Proof. See below. ■

Derivations and Proofs The Y_t process in Drechsler and Yaron (2011) is

$$\begin{pmatrix} \Delta c_{t+1} \\ x_{t+1} \\ \overline{\sigma}_{t+1}^2 \\ \sigma_{t+1}^2 \\ \Delta d_{t+1} \end{pmatrix} = \mu + F \begin{pmatrix} \Delta c_t \\ x_t \\ \overline{\sigma}_t^2 \\ \sigma_t^2 \\ \Delta d_t \end{pmatrix} + G_t \begin{pmatrix} z_{c,t+1} \\ z_{x,t+1} \\ z_{\overline{\sigma},t+1} \\ z_{\sigma,t+1} \\ z_{d,t+1} \end{pmatrix} + \begin{pmatrix} 0 \\ J_{x,t+1} \\ 0 \\ J_{\sigma,t+1} \\ 0 \end{pmatrix},$$

where the $z_{i,t+1}$ shocks are normally distributed, and

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \rho_x & 0 & 0 & 0 \\ 0 & 0 & \rho_{\overline{\sigma}} & 0 & 0 \\ 0 & 0 & (1-\widetilde{\rho}_\sigma) & \rho_\sigma & 0 \\ 0 & \phi & 0 & 0 & 0 \end{pmatrix}.$$

Drechsler and Yaron (2011) define G as

$$G_t G_t^{'} = h + H_\sigma \sigma_t^2,$$

with

$$h = \begin{pmatrix} h_c & 0 & 0 & 0 & \varphi_c\varphi_d\sqrt{1-\omega_c}\sqrt{1-\omega_d}\Omega_{cd} \\ 0 & h_x & 0 & 0 & 0 \\ 0 & 0 & h_{\overline{\sigma}} & 0 & 0 \\ 0 & 0 & 0 & h_\sigma & 0 \\ \varphi_c\varphi_d\sqrt{1-\omega_c}\sqrt{1-\omega_d}\Omega_{cd} & 0 & 0 & 0 & h_d \end{pmatrix}$$

and

$$H_\sigma = \begin{pmatrix} \overline{H}_c & 0 & 0 & 0 & \varphi_c\varphi_d\sqrt{\omega_c}\sqrt{\omega_d}\Omega_{cd} \\ 0 & \overline{H}_x & 0 & 0 & 0 \\ 0 & 0 & \overline{H}_{\overline{\sigma}} & 0 & 0 \\ 0 & 0 & 0 & \overline{H}_\sigma & 0 \\ \varphi_c\varphi_d\sqrt{\omega_c}\sqrt{\omega_d}\Omega_{cd} & 0 & 0 & 0 & \overline{H}_d \end{pmatrix}$$

with

$$h_i = \varphi_i^2 (1-\omega_i) \mathbb{E}\left[\sigma_t^2\right], \overline{H}_i = \varphi_i^2 \omega_i,$$

$J_{x,t}$ and $J_{\sigma,t}$ are compound Poisson processes defined as

$$J_{x,t+1} = \sum_{j=1}^{N_{t+1}^x} \xi_{j,t+1}^x \text{ where } N_{t+1}^x \sim \text{Poisson}\left(\lambda_t^x\right) \text{ and } \xi_{j,t+1}^x \sim -\Gamma\left(\nu_x, \frac{\mu_x}{\nu_x}\right) + \mu_x$$

and

$$J_{\sigma,t+1} = \sum_{j=1}^{N_{t+1}^\sigma} \xi_{j,t+1}^\sigma \text{ where } N_{t+1}^\sigma \sim \text{Poisson}\left(\lambda_t^\sigma\right) \text{ and } \xi_{j,t+1}^\sigma \sim \Gamma\left(\nu_\sigma, \frac{\mu_\sigma}{\nu_\sigma}\right)$$

where $\Gamma(x,y)$ represents a gamma distribution with shape parameter x and scale parameter y.[46] The jump intensities are

$$\lambda_t^x = l_0^x + l_{1,\sigma}^x \sigma_t^2 \text{ and } \lambda_t^\sigma = l_0^\sigma + l_{1,\sigma}^\sigma \sigma_t^2$$

where $l_0^\sigma = l_0^x = 0$. Note that in this model,

$$\rho_\sigma = \widetilde{\rho}_\sigma - l_{1,\sigma}^\sigma \mu_\sigma$$

and:

$$\begin{aligned}
\mathbb{E}x_{t+1} &= \frac{\widetilde{\mu}_x}{1-\rho_x} \\
\mathbb{E}\overline{\sigma}_{t+1}^2 &= \frac{\widetilde{\mu}_{\overline{\sigma}}}{1-\rho_{\overline{\sigma}}} \\
\mathbb{E}\sigma_{t+1}^2 &= \frac{\widetilde{\mu}_\sigma}{(1-\widetilde{\rho}_\sigma)} + \frac{\widetilde{\mu}_{\overline{\sigma}}}{(1-\rho_{\overline{\sigma}})}
\end{aligned}$$

with

$$\widetilde{\mu}_x = 0,\ \widetilde{\mu}_{\overline{\sigma}} = 1-\rho_{\overline{\sigma}},\ \widetilde{\mu}_\sigma = 0.$$

We now provide a proof of Result IA.7.

Proof. Proof of Result IA.7.Consider a jump in the state variable x :

$$J_{x,t+1} = \sum_{j=1}^{N_{t+1}^x} \left(-\xi_{j,t+1}^{x*} + \mu_x\right) = N_{t+1}^x \mu_x - \sum_{j=1}^{N_{t+1}^x} \xi_{j,t+1}^{x*}$$

[46] Drechsler and Yaron (2011) also explore using $\xi_{j,t+1}^x \sim N\left(0,\sigma_x^2\right)$ and $\xi_{j,t+1}^\sigma \sim N\left(0,\sigma_\sigma^2\right)$ as alternatives, which we omit here.

where

$$\xi_{j,t+1}^{x*} \sim \Gamma\left(\nu_x, \frac{\mu_x}{\nu_x}\right).$$

Thus

$$\begin{aligned}
\mathbb{E}_t\left[\exp\left\{u_k J_{x,t+1}\right\}\right] &= \mathbb{E}_t\left\{\mathbb{E}_t\left[\exp\left\{u_k J_{x,t+1}\right\}|N_{t+1}^x\right]\right\} \\
&= \mathbb{E}_t\left\{\mathbb{E}_t\left[\exp\left\{u_k N_{t+1}^x \mu_x - u_k \sum_{j=1}^{N_{t+1}^x} \xi_{j,t+1}^{x*}\right\}|N_{t+1}^x\right]\right\} \\
&= \mathbb{E}_t\left\{\exp\left(u_k N_{t+1}^x \mu_x\right)\mathbb{E}_t\left[\exp\left\{-u_k X\right\}\right]\right\},
\end{aligned}$$

where

$$X = \sum_{j=1}^{N_{t+1}^x} \xi_{j,t+1}^{x*}|N_{t+1}^x \sim \Gamma\left(N_{t+1}^x \nu_x, \frac{\mu_x}{\nu_x}\right).$$

Note

$$\mathbb{E}_t\left[\exp\left\{-u_k X\right\}\right] = \exp\left\{-N_{t+1}^x \nu_x \ln\left(1 + u_k \frac{\mu_x}{\nu_x}\right)\right\}.$$

Hence

$$\begin{aligned}
\mathbb{E}_t\left[\exp\left\{u_k J_{x,t+1}\right\}\right] &= \mathbb{E}_t\left\{\exp\left(u_k N_{t+1}^x \mu_x - N_{t+1}^x \nu_x \ln\left(1 + u_k \frac{\mu_x}{\nu_x}\right)\right)\right\} \\
&= \mathbb{E}_t\left\{\exp\left(N_{t+1}^x \Delta_k\right)\right\}
\end{aligned}$$

with

$$\Delta_k = u_k \mu_x - \nu_x \ln\left(1 + u_k \frac{\mu_x}{\nu_x}\right).$$

Thus,

$$\begin{aligned}
\mathbb{E}_t\left[\exp\left\{u_k J_{x,t+1}\right\}\right] &= \mathbb{E}_t\left\{\exp\left(N_{t+1}^x \Delta_k\right)\right\} \\
&= \exp\left(\lambda_t^x\left(e^{\Delta_k} - 1\right)\right) \\
&= \exp\left(\lambda_t^x\left(\psi\left[u_k\right] - 1\right)\right)
\end{aligned}$$

with

$$\psi\left[u_k\right] = \exp\left\{u_k \mu_x - \nu_x \ln\left(1 + u_k \frac{\mu_x}{\nu_x}\right)\right\}$$

Next, consider a jump in σ^2:

$$\begin{aligned}\mathbb{E}_t\left[\exp\left\{u_k J_{\sigma,t+1}\right\}\right] &= \mathbb{E}_t\left\{\mathbb{E}_t\left[\exp\left\{u_k J_{\sigma,t+1}\right\}|N_{t+1}^{\sigma}\right]\right\} \\ &= \mathbb{E}_t\left\{\mathbb{E}_t\left[\exp\left\{u_k\sum_{j=1}^{N_{t+1}^{\sigma}}\xi_{j,t+1}^{\sigma}\right\}|N_{t+1}^{\sigma}\right]\right\} \\ &= \mathbb{E}_t\left\{\mathbb{E}_t\left[\exp\left\{u_k X\right\}\right]\right\}\end{aligned}$$

where

$$X = \sum_{j=1}^{N_{t+1}^{\sigma}}\xi_{j,t+1}^{\sigma}|N_{t+1}^{\sigma} \sim \Gamma\left(N_{t+1}^{\sigma}\nu_{\sigma}, \frac{\mu_{\sigma}}{\nu_{\sigma}}\right)$$

Note

$$\mathbb{E}_t\left[\exp\left\{u_k X\right\}\right] = \exp\left\{-N_{t+1}^{\sigma}\nu_{\sigma}\ln\left(1-u_k\frac{\mu_{\sigma}}{\nu_{\sigma}}\right)\right\}.$$

Hence

$$\mathbb{E}_t\left[\exp\left\{u_k J_{\sigma,t+1}\right\}\right] = \mathbb{E}_t\left\{\exp\left(N_{t+1}^{\sigma}\Delta_k\right)\right\}$$

with

$$\Delta_k = -\nu_{\sigma}\ln\left(1-u_k\frac{\mu_{\sigma}}{\nu_{\sigma}}\right).$$

Thus,

$$\mathbb{E}_t\left[\exp\left\{u_k J_{\sigma,t+1}\right\}\right] = \exp\left(\lambda_t^{\sigma}\left(\psi\left[u_k\right]-1\right)\right)$$

with

$$\psi\left[u_k\right] = \exp\left\{-\nu_{\sigma}\ln\left(1-u_k\frac{\mu_{\sigma}}{\nu_{\sigma}}\right)\right\}.$$

In this model, the log-SDF is

$$m_{t+1} = \theta\log\delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)\, r_{c,t+1},$$

where

$$r_{c,t+1} = \kappa_0 + \kappa_1 \upsilon_{t+1} - \upsilon_t + \Delta c_{t+1}$$

with

$$\upsilon_t = A_0 + A^{'}Y_t,$$

where $A^{'} = (A_c, A_x, A_{\overline{\sigma}}, A_{\sigma}, A_d)$ and the market return is

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m}\upsilon_{m,t+1} - \upsilon_{m,t} + \Delta d_{t+1}$$

with

$$\upsilon_{m,t} = A_{0,m} + A_m^{'} Y_t$$

where $A_m = (A_{c,m}, A_{x,m}, A_{\overline{\sigma},m}, A_{\sigma,m}, A_{d,m})$.

Finding A_0, A, κ_0, and κ_1: In this model, we have

$$\begin{aligned} m_{t+1} + r_{c,t+1} &= \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta r_{c,t+1} \\ &= \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta \left(\kappa_0 + \kappa_1 \upsilon_{t+1} - \upsilon_t + \Delta c_{t+1} \right). \end{aligned}$$

Denote e_c the vector that selects consumption in Y_{t+1}, we have $\Delta c_{t+1} = e_c^{'} Y_{t+1}$. Thus,

$$\begin{aligned} m_{t+1} + r_{c,t+1} &= \theta \log \delta + \theta \kappa_0 + (\kappa_1 - 1) \theta A_0 - \theta A^{'} Y_t \\ &\quad + \left(\left(\theta - \frac{\theta}{\psi} \right) e_c^{'} + \theta \kappa_1 A^{'} \right) Y_{t+1} \end{aligned}$$

Denote

$$u^{'} = \left(\theta - \frac{\theta}{\psi} \right) e_c^{'} + \theta \kappa_1 A^{'}.$$

Then

$$\mathbb{E}_t e^{(m_{t+1} + r_{c,t+1})} = \left\{ \exp \left(\theta \log \delta + \theta \kappa_0 + (\kappa_1 - 1) \theta A_0 - \theta A^{'} Y_t \right) \right\} \times \left\{ \mathbb{E}_t \exp \left(u^{'} Y_{t+1} \right) \right\}$$

From Equation A.1.2 in Drechsler and Yaron (2011), we have

$$\mathbb{E}_t \left\{ \exp \left(u^{'} Y_{t+1} \right) | Y_t \right\} = \exp \left(\mathbf{f}(u) + \mathbf{g}(u)^{'} Y_t \right)$$

with

$$\begin{aligned} \mathbf{f}(u) &= \mu^{'} u + \frac{1}{2} u^{'} h u + l_0^{'} \left(\psi(u) - 1 \right), \\ \mathbf{g}(u) &= F^{'} u + \frac{1}{2} \left[u^{'} H_i u \right]_{i \in \{1, \ldots, n\}} + l_1^{'} \left(\psi(u) - 1 \right), \end{aligned}$$

where $\left[u^{'}H_iu\right]_{i\in\{1,\ldots,n\}}$ is the $n\times 1$ vector with i component equal to $u^{'}H_iu$. Thus,

$$\mathbb{E}_t e^{(m_{t+1}+r_{c,t+1})} = \exp\left(\begin{array}{c}\theta\log\delta+\theta\kappa_0+(\kappa_1-1)\,\theta A_0+\mathbf{f}\,(u)\\ +\left(\mathbf{g}\,(u)^{'}-\theta A^{'}\right)Y_t\end{array}\right).$$

The representative agent's first-order conditions imply

$$\log\mathbb{E}_t e^{(m_{t+1}+r_{c,t+1})}=0$$

which implies

$$\theta\log\delta+\theta\kappa_0+(\kappa_1-1)\,\theta A_0+\mathbf{f}\,(u)+\left(\mathbf{g}\,(u)^{'}-\theta A^{'}\right)Y_t=0$$

This implies that

$$\begin{cases}\mathbf{g}\,(u)^{'}-\theta A^{'}=0\\ \theta\log\delta+\theta\kappa_0+(\kappa_1-1)\,\theta A_0+\mathbf{f}\,(u)=0\end{cases}$$

This system of equations is a function of κ_0 and κ_1. These two coefficients are solved using

$$\begin{aligned}\kappa_0 &= -\kappa_1\log\kappa_1-(1-\kappa_1)\log(1-\kappa_1)\\ \kappa_0+(\kappa_1-1)\,A_0 &= -\log\kappa_1+(1-\kappa_1)\,A^{'}\mathbb{E}\left[Y_t\right]\end{aligned}$$

Together A_0, A, κ_0, κ_1 can be solved by using the system of equations

$$\begin{cases}\mathbf{g}\,(u)^{'}-\theta A^{'}=0,\\ \theta\log\delta+\theta\kappa_0+(\kappa_1-1)\,\theta A_0+\mathbf{f}\,(u)=0,\\ -\kappa_0-\kappa_1\log\kappa_1-(1-\kappa_1)\log(1-\kappa_1)=0,\\ \kappa_0+(\kappa_1-1)\,A_0+\log\kappa_1-(1-\kappa_1)\,A^{'}\mathbb{E}\left[Y_t\right]=0.\end{cases}$$

Finding $A_{0,m}$, A_m, $\kappa_{0,m}$, and $\kappa_{1,m}$: Recall that

$$\begin{aligned} m_{t+1}+r_{m,t+1} &= \theta\log\delta-\frac{\theta}{\psi}\Delta c_{t+1}+(\theta-1)\,r_{c,t+1} \\ &\quad+\kappa_{0,m}+\kappa_{1,m}\upsilon_{m,t+1}-\upsilon_{m,t}+\Delta d_{t+1}. \end{aligned}$$

Note that

$$\begin{aligned} m_{t+1}+r_{m,t+1} &= \theta\log\delta-\frac{\theta}{\psi}\Delta c_{t+1}+(\theta-1)\,r_{c,t+1}+r_{m,t+1} \\ &= \mathbb{B}_0+\mathbb{B}_2^{'}Y_{t+1}+\mathbb{B}_1^{'}Y_t \end{aligned}$$

where

$$\begin{aligned} \mathbb{B}_0 &= \theta\log\delta+(\kappa_1-1)(\theta-1)A_0+(\theta-1)\kappa_0+\kappa_{0,m}+(\kappa_{1,m}-1)A_{0,m}, \\ \mathbb{B}_1 &= (1-\theta)A-A_m, \\ \mathbb{B}_2 &= -\Lambda+e_d+\kappa_{1,m}A_m, \end{aligned}$$

and

$$\Lambda=\gamma e_c+(1-\theta)\kappa_1 A.$$

Thus, $\log\mathbb{E}_t e^{(m_{t+1}+r_{m,t+1})}=0$ implies

$$\mathbb{B}_0+\mathbb{B}_1^{'}Y_t+\mathbf{f}(\mathbb{B}_2)+\mathbf{g}(\mathbb{B}_2)^{'}Y_t=0,$$

which implies that

$$\begin{cases} \mathbb{B}_0+\mathbf{f}(\mathbb{B}_2)=0, \\ \\ \mathbb{B}_1+\mathbf{g}(\mathbb{B}_2)=0. \end{cases}$$

These two equations depend on $\kappa_{0,m}$ and $\kappa_{1,m}$, which can be obtained from the two equations below

$$-\kappa_{0,m}-\kappa_{1,m}\log\kappa_{1,m}-(1-\kappa_{1,m})\log(1-\kappa_{1,m})=0,$$

$$\kappa_{0,m}+(\kappa_{1,m}-1)A_{0,m}+\log\kappa_{1,m}-(1-\kappa_{1,m})A_m^{'}\mathbb{E}[Y_t]=0.$$

Finally $A_{0,m}$, A_m, $\kappa_{0,m}$, $\kappa_{1,m}$ can be obtained by simultaneously solving the four equations below:

$$\begin{cases} \left\{ \begin{array}{c} \theta \log \delta + (\kappa_1 - 1)(\theta - 1) A_0 + (\theta - 1)\kappa_0 + \kappa_{0,m} \\ + (\kappa_{1,m} - 1) A_{0,m} + \mathbf{f}(-\Lambda + e_d + \kappa_{1,m} A_m) \end{array} \right\} = 0, \\ (1-\theta) A - A_m + \mathbf{g}(-\Lambda + e_d + \kappa_{1,m} A_m) = 0, \\ -\kappa_{0,m} - \kappa_{1,m} \log \kappa_{1,m} - (1 - \kappa_{1,m}) \log (1 - \kappa_{1,m}) = 0, \\ \kappa_{0,m} + (\kappa_{1,m} - 1) A_{0,m} + \log \kappa_{1,m} - (1 - \kappa_{1,m}) A_m^{'} \mathbb{E}[Y_t] = 0. \end{cases}$$

Now let us find the risk-free rate

$$\log R_{f,t\to t+1} = -\log \left(\mathbb{E}_t e^{m_{t+1}} \right).$$

Note that

$$\begin{aligned} m_{t+1} &= \theta \log \delta - \frac{\theta}{\psi} e_c^{'} Y_{t+1} + (\theta - 1)\left(\kappa_0 + \kappa_1 \upsilon_{t+1} - \upsilon_t + e_c^{'} Y_{t+1} \right) \\ &= \theta \log \delta - \frac{\theta}{\psi} e_c^{'} Y_{t+1} + (\theta - 1)\kappa_0 + (\theta - 1)\kappa_1 \left(A_0 + A^{'} Y_{t+1} \right) \\ &\quad - (\theta - 1)\left(A_0 + A^{'} Y_t \right) + (\theta - 1) e_c^{'} Y_{t+1} \end{aligned}$$

which simplifies to

$$\begin{aligned} m_{t+1} &= \{\theta \log \delta + (\theta - 1)\kappa_0 + (\theta - 1)\kappa_1 A_0 - (\theta - 1) A_0\} \\ &\quad + \left\{ \left(-\frac{\theta}{\psi} + (\theta - 1) \right) e_c^{'} + (\theta - 1)\kappa_1 A^{'} \right\} Y_{t+1} \\ &\quad - (\theta - 1) A^{'} Y_t \end{aligned}$$

and

$$m_{t+1} = \left\{ \theta \log \delta - (1-\theta)[\kappa_0 + A_0(\kappa_1 - 1)] - (\theta - 1) A^{'} Y_t \right\} - \Lambda^{'} Y_{t+1}.$$

Thus,

$$\begin{aligned} -\log R_{f,t\to t+1} &= \theta \log \delta - (1-\theta)[\kappa_0 + A_0(\kappa_1 - 1)] - (\theta - 1) A^{'} Y_t \\ &\quad + \mathbf{f}(-\Lambda) + \mathbf{g}(-\Lambda)^{'} Y_t \end{aligned}$$

which simplifies to:

$$\log R_{f,t\to t+1} = -\left\{\begin{array}{c}\theta\log\delta + \mathbf{f}\left(-\Lambda\right)\\ -\left(1-\theta\right)\left[\kappa_0 + A_0\left(\kappa_1 - 1\right)\right]\end{array}\right\} - \left(\left(1-\theta\right)A + \mathbf{g}\left(-\Lambda\right)\right)^{'}Y_t.$$

■

We now provide a proof of Result IA.8.

Proof. Proof of Result IA.8.

Expression for $\mathbb{E}_t\left[\boldsymbol{R}^{n}_{\boldsymbol{M,t\to t+1}}\right]$:

$$\mathbb{E}_t R^n_{M,t\to t+1} = \mathbb{E}_t e^{nr_{m,t+1}}$$

Observe that

$$\begin{aligned} nr_{m,t+1} &= n\kappa_{0,m} + n\kappa_{1,m}\upsilon_{m,t+1} - n\upsilon_{m,t} + ne_d^{'}Y_{t+1} \\ &= \left(n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}\right) - nA_m^{'}Y_t \\ &\quad + \left(n\kappa_{1,m}A_m + ne_d\right)^{'}Y_{t+1}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}_t R^n_{M,t\to t+1} &= \exp\left\{\left(n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}\right) - nA_m^{'}Y_t\right\} \\ &\quad \times\mathbb{E}_t\exp\left\{\left(n\kappa_{1,m}A_m + ne_d\right)^{'}Y_{t+1}\right\} \end{aligned}$$

and

$$\mathbb{E}_t R^n_{M,t\to t+1} = \exp\left\{\begin{array}{c}\left(n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}\right) + \mathbf{f}\left(n\kappa_{1,m}A_m + ne_d\right)\\ +\left\{\mathbf{g}\left(n\kappa_{1,m}A_m + ne_d\right) - nA_m\right\}^{'}Y_t\end{array}\right\}.$$

The truncated physical moment is

$$\mathbb{E}_t\left[R^n_{M,t+1}1_{R_{M,t+1}>a}\right] = \mathbb{E}_t\left\{\left(\exp\left\{\mathcal{A}_{y,t} + \left(\Lambda^* + \Lambda\right)^{'}Y_{t+1}\right\}\right)\mathbb{I}_{\left\{\mathcal{A}_{y,t}+\left(\Lambda^*+\Lambda\right)^{'}Y_{t+1}>n\log a\right\}}\right\}$$

where

$$\mathcal{A}_{y,t} = \left(n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}\right) - nA_m^{'}Y_t.$$

■

We now provide a proof of Result IA.9.

Proof. Proof of Result IA.9.

Expression for $\mathbb{E}_t^*\left[\boldsymbol{R}^n_{M,t\to t+1}\right]$:

$$\mathbb{E}_t^* R^n_{M,t\to t+1} = R_{f,t+1}\mathbb{E}_t e^{m_{t+1}+nr_{m,t+1}}.$$

Observe that

$$\begin{aligned} m_{t+1} + nr_{m,t+1} &= \left\{\theta\log\delta - (1-\theta)\left[\kappa_0 + A_0(\kappa_1 - 1)\right] - (\theta-1)A'Y_t\right\} - \Lambda' Y_{t+1} \\ &\quad + (n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}) - nA_m' Y_t + (n\kappa_{1,m}A_m + ne_d)' Y_{t+1} \end{aligned}$$

which simplifies to

$$\begin{aligned} m_{t+1} + nr_{m,t+1} &= \left\{\begin{array}{c} \theta\log\delta - (1-\theta)\left[\kappa_0 + A_0(\kappa_1 - 1)\right] \\ + (n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}) \\ -(\theta-1)A'Y_t - nA_m' Y_t \end{array}\right\} \\ &\quad + \left\{n\kappa_{1,m}A_m + ne_d - \Lambda\right\}' Y_{t+1}. \end{aligned}$$

Denote

$$\Lambda^* = n\kappa_{1,m}A_m + ne_d - \Lambda.$$

Thus

$$m_{t+1} + nr_{m,t+1} = \left\{\begin{array}{c} \theta\log\delta - (1-\theta)\left[\kappa_0 + A_0(\kappa_1 - 1)\right] \\ + (n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}) \\ -(\theta-1)A'Y_t - nA_m' Y_t \end{array}\right\} + \Lambda^{*'} Y_{t+1}.$$

Therefore,

$$\mathbb{E}_t^* R^n_{M,t\to t+1} = R_{f,t+1}\exp\left\{\begin{array}{c} \theta\log\delta - (1-\theta)\left[\kappa_0 + A_0(\kappa_1 - 1)\right] \\ + (n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}) + \mathbf{f}(\Lambda^*) \\ \left\{-(\theta-1)A - nA_m + \mathbf{g}(\Lambda^*)\right\} Y_t \end{array}\right\}.$$

The truncated risk neutral moment is

$$\mathbb{E}_t^*\left[R_{M,t+1}^n 1_{R_{M,t+1}>a}\right] = R_{f,t+1}\mathbb{E}_t^*\left[e^{m_{t+1}+nr_{m,t+1}} 1_{R_{M,t+1}>a}\right]$$

Note that

$$m_{t+1} + nr_{m,t+1} = \mathcal{A}_{y,t}^* + \Lambda^{*'} Y_{t+1}$$

where

$$\mathcal{A}_{y,t}^* = \left\{\begin{array}{c} \theta \log \delta - (1-\theta)\left[\kappa_0 + A_0\left(\kappa_1 - 1\right)\right] \\ \\ + \left(n\kappa_{0,m} + n\kappa_{1,m}A_{0,m} - nA_{0,m}\right) \\ \\ -\left(\theta - 1\right) A' Y_t - nA_m' Y_t \end{array}\right\}.$$

Therefore,

$$\mathbb{E}_t^*\left[R_{M,t+1}^n 1_{R_{M,t+1}>a}\right] = R_{f,t+1}\mathbb{E}_t^*\left[e^{\mathcal{A}_{y,t}^*+\Lambda^{*'}Y_{t+1}} 1_{\mathcal{A}_{y,t}+(\Lambda^*+\Lambda)' Y_{t+1}>n \log a}\right].$$

■

IA.7.4 Habit Formation Models: Bekaert, Engstrom, and Ermolov (2020)

Bekaert, Engstrom, and Ermolov (2020) consider an expected utility function of the form

$$\mathbb{E}_t\left[\sum_{j=t}^{\infty} \beta^{j-t} \frac{\left(C_j - H_j\right)^{1-\gamma} - 1}{1-\gamma}\right],$$

where β is the time discount rate, C_j is consumption and H_j is the habit stock with the restriction $C_j > H_j$. In this framework, the log-SDF is given by

$$m_{t\to t+1} = \log \beta - \gamma g_{t+1} + \gamma \Delta q_{t+1},$$

where $g_{t+1} \equiv \log\left(C_{t+1}/C_t\right)$ is the log of consumption growth, and $q_t = \log Q_t$ with $Q_t \equiv C_t/\left(C_t - H_t\right)$. We consider the two models studied in Bekaert, Engstrom, and Ermolov (2020): (i) their model without preference shocks,[47] and (ii) their model with preference shocks. While the model without preference shocks is able to explain many standard stylized facts in the data, it falls short of explaining the low variance risk premium persistence. The model with preference shocks is designed to simultaneously explain the low variance risk premium persistence and the average variance risk premium itself. We present

[47] We use their "Full" model specification in this case, as opposed to their "Baseline" specification.

here a general specification that allows for preference shocks then later specialize it by setting some coefficients to zero to obtain the model without preference shocks.

Bekaert, Engstrom, and Ermolov (2020) consider an economy having the following time-series dynamics:

$$\begin{aligned}
g_{t+1} &= \overline{g} + \phi_g \left(n_t - \overline{n}\right) + \sigma_{cp}\omega_{p,t+1} - \sigma_{cn}\omega_{n,t+1}, \\
d_{t+1} &= \overline{g} + \phi_d \left(n_t - \overline{n}\right) + \gamma_g \left(\sigma_{cp}\omega_{p,t+1} - \sigma_{cn}\omega_{n,t+1}\right) + \gamma_n \left(-\sigma_{cn}\omega_{n,t+1}\right), \\
q_{t+1} &= \overline{q} + \rho_q \left(q_t - \overline{q}\right) + \sigma_{qp}\omega_{p,t+1} + \sigma_{qn}\omega_{n,t+1} + \sigma_{qq}\omega_{q,t+1}.
\end{aligned}$$

where

$$\begin{aligned}
p_{t+1} &= \overline{p} + \rho_p \left(p_t - \overline{p}\right) + \sigma_{pp}\omega_{p,t+1}, \\
n_{t+1} &= \overline{n} + \rho_n \left(n_t - \overline{n}\right) + \sigma_{nn}\omega_{n,t+1}, \\
s_{t+1} &= \overline{s} + \rho_s \left(s_t - \overline{s}\right) + \sigma_{sq}\omega_{q,t+1},
\end{aligned}$$

and where $\omega_{p,t+1}$, $\omega_{n,t+1}$, $\omega_{q,t+1}$ follow demeaned gamma distributions defined by

$$\begin{aligned}
\omega_{p,t+1} &\sim \Gamma\left(p_t, 1\right) - p_t, \\
\omega_{n,t+1} &\sim \Gamma\left(n_t, 1\right) - n_t, \\
\omega_{q,t+1} &\sim \Gamma\left(s_t, 1\right) - s_t.
\end{aligned}$$

Here, $\Gamma\left(x, y\right)$ is a gamma distribution with shape parameter x and scale parameter y. $\omega_{p,t+1}$, $\omega_{n,t+1}$, and $\omega_{q,t+1}$ are independent and have zero mean. The log-SDF has the form

$$m_{t\to t+1} = m_0 + m_q q_t + m_n n_t + m_{\omega,p}\omega_{p,t+1} + m_{\omega,n}\omega_{n,t+1} + m_{\omega,q}\omega_{q,t+1},$$

See below for more details on parameters in the log-SDF. There are, in general, three state variables in this framework: q_t, n_t, and s_t. The model without preference shocks obtains when the state variable s_t is set to zero and certain parameter restrictions are imposed (see below for more details). Given this setup, we show the following results.

Main Results

Result IA.10. *Given $p_t = \overline{p}$ and the state variables q_t, n_t, and s_t, the Bekaert, Engstrom, and Ermolov (2020) model-implied log price-dividend ratio is given by*

$$\log \frac{P_t}{D_t} = K_0^1 + K_p^1 p_t + K_n^1 n_t + K_q^1 q_t + K_s^1 s_t \qquad \text{(IA.77)}$$

and the non-central risk-neutral market return moments are given by

$$\mathbb{E}_t^*\left[R_{M,t\to t+1}^n\right] = R_{f,t\to t+1}\exp\left(\begin{array}{c} nr_0 + m_0 + \left\{nr_p - g\left(nr_{\omega p} + m_{\omega,p}\right)\right\}p_t \\ + \left\{nr_n + m_n - g\left(m_{\omega,n} + nr_{\omega n}\right)\right\}n_t \\ + \left(nr_q + m_q\right)q_t + \left(nr_s - g\left(nr_{\omega q} + m_{\omega,q}\right)\right)s_t \end{array}\right) \quad \text{(IA.78)}$$

with

$$R_{f,t\to t+1} = \exp\left(f_0 + f_q q_t + f_n n_t + f_p p_t + f_s s_t\right).$$

All parameters are described in more detail below. The function $g(\cdot)$ is defined as $g(x) = x + \log(1-x)$. $R_{f,t\to t+1}$ is the model-implied risk-free rate. Given $\mathbb{E}_t^*\left[R_{M,t\to t+1}^k\right]$, $\mathbb{M}_{t\to t+1}^{*(n)}$ can be computed using Equation 17 by setting $x = R_{M,t\to t+1}$ and $x_s = R_{f,t\to t+1}$, then taking expectations under the risk-neutral measure. The resulting expression can be found in Equation IA.76.

Proof. See below. ■

Result IA.11. *The conditional non-central physical moment of the market return is*

$$\mathbb{E}_t\left[R_{M,t\to t+1}^k\right] = \exp\left\{\begin{array}{c} kr_0 + kr_p p_t + kr_n n_t + kr_q q_t + kr_s s_t \\ -p_t g\left(kr_{\omega p}\right) - n_t g\left(kr_{\omega n}\right) - s_t g\left(kr_{\omega q}\right) \end{array}\right\}.$$

The truncated non-central physical moment is

$$\mathbb{E}_t\left[R_{M,t\to t+1}^k \mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] = e^{\xi_{r,t}}\mathbb{E}_t\left[e^{Z_{t+1,r}}\mathbb{I}_{\{Z_{t+1,r}>k\ln a-\xi_{r,t}\}}\right]$$

with

$$\begin{aligned} \xi_{r,t} &= kr_0 + \left(kr_p - kr_{\omega p}\right)p_t + \left(kr_n - kr_{\omega n}\right)n_t + kr_q q_t + \left(kr_s - kr_{\omega q}\right)s_t \\ Z_{t+1,r} &= kr_{\omega p}\left(\omega_{p,t+1} + p_t\right) + kr_{\omega n}\left(\omega_{n,t+1} + n_t\right) + kr_{\omega q}\left(\omega_{q,t+1} + s_t\right). \end{aligned}$$

with $p_t = \overline{p}$ and

$$\begin{aligned} \omega_{p,t+1} + p_t &\sim \Gamma\left(p_t, 1\right), \\ \omega_{n,t+1} + n_t &\sim \Gamma\left(n_t, 1\right), \\ \omega_{q,t+1} + s_t &\sim \Gamma\left(s_t, 1\right), \end{aligned}$$

Given the state variables, the truncated moment $\mathbb{E}_t\left[R_{M,t\to t+1}^k \mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right]$ can be computed by simulation at each time t. All parameters are defined below.

Proof. See below. ■

Result IA.12. *The conditional non-central risk neutral moment of the market return is*

$$\mathbb{E}_t^*\left[R_{M,t\to t+1}^k\right] = R_{f,t\to t+1}\exp\left(\begin{array}{c} kr_0 + m_0 + \left\{kr_p - g\left(kr_{\omega p} + m_{\omega,p}\right)\right\}p_t \\ + \left\{kr_n + m_n - g\left(m_{\omega,n} + kr_{\omega n}\right)\right\}n_t \\ + \left(kr_q + m_q\right)q_t + \left(kr_s - g\left(kr_{\omega q} + m_{\omega,q}\right)\right)s_t \end{array}\right).$$

The truncated non-central risk neutral moment is

$$\mathbb{E}_t^*\left[R_{M,t\to t+1}^k \mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] = R_{f,t+1}\mathbb{E}_t\left[e^{\zeta_r + Z_{t+1,m}} 1_{Z_{t+1,r}>k\log a - \xi_{r,t}}\right].$$

where

$$\begin{aligned}
\zeta_r &= m_0 + kr_0 + \left(kr_p - kr_{\omega p} - m_{\omega,p}\right)p_t + \left(kr_n - kr_{\omega n} + m_n - m_{\omega,n}\right)n_t \\
&\quad + \left(kr_s - kr_{\omega q} - m_{\omega,q}\right)s_t + \left(kr_q + m_q\right)q_t, \\
Z_{t+1,m} &= \left(m_{\omega,p} + kr_{\omega p}\right)\left(\omega_{p,t+1} + p_t\right) + \left(m_{\omega,n} + kr_{\omega n}\right)\left(\omega_{n,t+1} + n_t\right) \\
&\quad + \left(kr_{\omega q} + m_{\omega,q}\right)\left(\omega_{q,t+1} + s_t\right), \\
\xi_{r,t} &= kr_0 + \left(kr_p - kr_{\omega p}\right)p_t + \left(kr_n - kr_{\omega n}\right)n_t + kr_q q_t + \left(kr_s - kr_{\omega q}\right)s_t \\
Z_{t+1,r} &= kr_{\omega p}\left(\omega_{p,t+1} + p_t\right) + kr_{\omega n}\left(\omega_{n,t+1} + n_t\right) + kr_{\omega q}\left(\omega_{q,t+1} + s_t\right),
\end{aligned}$$

$$\omega_{p,t+1} + p_t \sim \Gamma\left(p_t, 1\right),\ \omega_{n,t+1} + n_t \sim \Gamma\left(n_t, 1\right),\ \textit{and}\ \omega_{q,t+1} + s_t \sim \Gamma\left(s_t, 1\right).$$

Given the state variables, $\mathbb{E}_t\left[e^{\zeta_r + Z_{t+1,m}} 1_{Z_{t+1,r}>k\log a - \xi_{r,t}}\right]$ *can be computed by simulation at each time* t*. All parameters are defined below.*

Proof. See below. ■

Derivations and Proofs We provide proofs of Results IA.10, IA.11, and IA.12 below.

Proof. Proofs of Results IA.10, IA.11, and IA.12.

Let us denote $r_{t+1} = \log R_{M,t\to t+1}$. The log-SDF with preference shocks

$$m_{t+1} = \log\beta - \gamma g_{t+1} + \gamma\Delta q_{t+1}$$

can be written as

$$\begin{aligned}
m_{t+1} &= \log\beta - \gamma q_t - \gamma g_{t+1} + \gamma q_{t+1} \\
&= \left\{\log\beta - \gamma\overline{g} + \gamma\phi_g\overline{n} + \gamma\overline{q}\left(1-\rho_q\right)\right\} + \left(\rho_q - 1\right)\gamma q_t - \gamma\phi_g n_t \\
&\quad + \left(\gamma\sigma_{qp} - \gamma\sigma_{cp}\right)\omega_{p,t+1} + \left(\gamma\sigma_{cn} + \gamma\sigma_{qn}\right)\omega_{n,t+1} + \gamma\sigma_{qq}\omega_{q,t+1}
\end{aligned}$$

This log-SDF simplifies to

$$\begin{aligned}
m_{t+1} &= \log\beta - \gamma q_t - \gamma g_{t+1} + \gamma q_{t+1} \\
&= m_0 + m_q q_t + m_n n_t + m_{\omega,p}\omega_{p,t+1} + m_{\omega,n}\omega_{n,t+1} + m_{\omega,q}\omega_{q,t+1}
\end{aligned}$$

with

$$
\begin{aligned}
m_0 &= \log \beta + \gamma\left(\phi_g \overline{n} - \overline{g}\right) - \overline{q} m_q, & m_n &= -\gamma \phi_g, \\
m_q &= -\left(1-\rho_q\right)\gamma, & m_{\omega,q} &= \gamma \sigma_{qq}, \\
m_{\omega,p} &= \gamma\left(\sigma_{qp} - \sigma_{cp}\right), & m_{\omega,n} &= \gamma\left(\sigma_{cn} + \sigma_{qn}\right).
\end{aligned}
$$

The log risk-free rate is

$$
\log R_{f,t\to t+1} = f_0 + f_q q_t + f_n n_t + f_p p_t + f_s s_t \tag{IA.79}
$$

with

$$
f_0 = -m_0,\ f_q = -m_q,\ f_n = -\left(m_n - g\left(m_{\omega,n}\right)\right),\ f_p = g\left(m_{\omega,p}\right),\ f_s = g\left(m_{\omega,q}\right).
$$

Now the price-dividend ratio is

$$
\begin{aligned}
\frac{P_t}{D_t} &= \mathbb{E}_t\left[e^{m_{t+1}+d_{t+1}}\right] + \mathbb{E}_t\left[e^{\sum_{j=1}^{2}\left(m_{t+j}+d_{t+j}\right)}\right] + \mathbb{E}_t\left[e^{\sum_{j=1}^{3}\left(m_{t+j}+d_{t+j}\right)}\right] + ... \\
&= \mathbb{E}_t\left[e^{m_{t+1}+d_{t+1}}\right] + \mathbb{E}_t\left[e^{m_{t+1}+d_{t+1}} e^{\left(m_{t+2}+d_{t+2}\right)}\right] + \mathbb{E}_t\left[e^{\sum_{j=1}^{3}\left(m_{t+j}+d_{t+j}\right)}\right] + ...
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& m_{t+1} + d_{t+1} \\
= \; & \left\{m_0 + \overline{g} - \phi_d \overline{n}\right\} + m_q q_t + \left\{m_n + \phi_d\right\} n_t \\
& + \left\{m_{\omega,p} + \gamma_g \sigma_{cp}\right\} \omega_{p,t+1} + \left\{m_{\omega,n} - \left(\gamma_n + \gamma_g\right)\sigma_{cn}\right\} \omega_{n,t+1} + m_{\omega,q}\omega_{q,t+1}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \mathbb{E}_t\left[e^{m_{t+1}+d_{t+1}}\right] \\
= \; & e^{\left\{\left\{m_0+\overline{g}-\phi_d\overline{n}\right\}+m_q q_t+\left\{m_n+\phi_d\right\}n_t\right\}} \\
& \times \mathbb{E}_t\left[e^{\left(m_{\omega,p}+\gamma_g\sigma_{cp}\right)\omega_{p,t+1}}\right] \mathbb{E}_t\left[e^{\left(m_{\omega,n}-\left(\gamma_n+\gamma_g\right)\sigma_{cn}\right)\omega_{n,t+1}}\right] \mathbb{E}_t\left[e^{m_{\omega,q}\omega_{q,t+1}}\right]
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\mathbb{E}_t\left[e^{\left(m_{\omega,p}+\gamma_g\sigma_{cp}\right)\omega_{p,t+1}}\right] &= \exp\left\{-p_t g\left(m_{\omega,p} + \gamma_g \sigma_{cp}\right)\right\}, \\
\mathbb{E}_t\left[e^{\left(m_{\omega,n}-\gamma_g\sigma_{cn}-\gamma_n\sigma_{cn}\right)\omega_{n,t+1}}\right] &= \exp\left\{-n_t g\left(m_{\omega,n} - \left(\gamma_g + \gamma_n\right)\sigma_{cn}\right)\right\}, \\
\mathbb{E}_t\left[e^{m_{\omega,q}\omega_{q,t+1}}\right] &= \exp\left\{-s_t g\left(m_{\omega,q}\right)\right\}.
\end{aligned}
$$

Finally,

$$\begin{aligned}
\mathbb{E}_t\left[e^{m_{t+1}+d_{t+1}}\right] &= \exp\left\{\begin{array}{c} \{m_0+\overline{g}-\phi_d\overline{n}\}+m_q q_t+\{m_n+\phi_d\}\,n_t-p_t g\left(m_{\omega,p}+\gamma_g\sigma_{cp}\right) \\ \\ -n_t g\left(m_{\omega,n}-\left(\gamma_g+\gamma_n\right)\sigma_{cn}\right)-s_t g\left(m_{\omega,q}\right) \end{array}\right\} \\
&= \exp\left\{A_1+B_1 p_t+C_1 n_t+D_1 q_t+G_1 s_t\right\}
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \overline{g}+m_0-\phi_d\overline{n}, \\
B_1 &= -g\left(m_{\omega,p}+\gamma_g\sigma_{cp}\right), \\
C_1 &= m_n+\phi_d-g\left(m_{\omega,n}-\left(\gamma_g+\gamma_n\right)\sigma_{cn}\right), \\
D_1 &= m_q, \\
G_1 &= -g\left(m_{\omega,q}\right).
\end{aligned}$$

Now, observe that

$$\mathbb{E}_{t+1}\left[e^{m_{t+2}+d_{t+2}}\right]=\exp\left\{A_1+B_1 p_{t+1}+C_1 n_{t+1}+D_1 q_{t+1}+G_1 s_{t+1}\right\},$$

and

$$\mathbb{E}_t\left[e^{m_{t+2}+d_{t+2}}\right]=\mathbb{E}_t\left[\exp\left\{A_1+B_1 p_{t+1}+C_1 n_{t+1}+D_1 q_{t+1}+G_1 s_{t+1}\right\}\right].$$

Recall that

$$\begin{aligned}
q_{t+1} &= \overline{q}+\rho_q\left(q_t-\overline{q}\right)+\sigma_{qp}\omega_{p,t+1}+\sigma_{qn}\omega_{n,t+1}+\sigma_{qq}\omega_{q,t+1}, \\
p_{t+1} &= \overline{p}+\rho_p\left(p_t-\overline{p}\right)+\sigma_{pp}\omega_{p,t+1}, \\
n_{t+1} &= \overline{n}+\rho_n\left(n_t-\overline{n}\right)+\sigma_{nn}\omega_{n,t+1}.
\end{aligned}$$

Thus:

$$\mathbb{E}_t\left[e^{m_{t+1}+d_{t+1}}\right]\mathbb{E}_{t+1}\left[e^{m_{t+2}+d_{t+2}}\right]=\mathbb{E}_t\left[e^{m_{t+1}+d_{t+1}}\right]\exp\left\{A_1+B_1 p_{t+1}+C_1 n_{t+1}+D_1 q_{t+1}+G_1 s_{t+1}\right\}.$$

However, observe that

$$e^{m_{t+1}+d_{t+1}}=e^{\{\{m_0+\overline{g}-\phi_d\overline{n}\}+m_q q_t+\{m_n+\phi_d\}n_t\}}\times e^{(m_{\omega,p}+\gamma_g\sigma_{cp})\omega_{p,t+1}}e^{(m_{\omega,n}-(\gamma_n+\gamma_g)\sigma_{cn})\omega_{n,t+1}}e^{m_{\omega,q}\omega_{q,t+1}}.$$

Thus:

$$\mathbb{E}_t\left[e^{m_{t+1}+d_{t+1}}\right]\mathbb{E}_{t+1}\left[e^{m_{t+2}+d_{t+2}}\right]=\mathbb{E}_t\left[\exp\left\{\begin{array}{c} \left(m_0+\overline{g}-\phi_d\overline{n}\right)+m_q q_t+\left(m_n+\phi_d\right)n_t \\ \\ A_1+B_1 p_{t+1}+C_1 n_{t+1}+D_1 q_{t+1}+G_1 s_{t+1} \\ \\ +\left(m_{\omega,p}+\gamma_g\sigma_{cp}\right)\omega_{p,t+1}+\left(m_{\omega,n}-\left(\gamma_n+\gamma_g\right)\sigma_{cn}\right)\omega_{n,t+1}+m_{\omega,q}\omega_{q,t+1} \end{array}\right\}\right]$$

which simplifies to

$$
\begin{aligned}
& \mathbb{E}_t\left[e^{m_{t+1}+d_{t+1}}\right]\mathbb{E}_{t+1}\left[e^{m_{t+2}+d_{t+2}}\right] \\
= \quad & \mathbb{E}_t\left[\exp\left\{\begin{array}{c}
A_2 \\
+\left(D_1\rho_q+m_q\right)q_t+\left(m_n+\phi_d+C_1\rho_n\right)n_t \\
+\left\{\begin{array}{c} B_1\sigma_{pp}+D_1\sigma_{qp}+\left(m_{\omega,p}+\gamma_g\sigma_{cp}\right)\end{array}\right\}\omega_{p,t+1}+B_1\rho_p p_t \\
+\left\{\begin{array}{c} C_1\sigma_{nn}+D_1\sigma_{qn}+\left(m_{\omega,n}-\left(\gamma_n+\gamma_g\right)\sigma_{cn}\right)\end{array}\right\}\omega_{n,t+1} \\
+\left\{D_1\sigma_{qq}+G_1\sigma_{sq}+m_{\omega,q}\right\}\omega_{q,t+1}+G_1\rho_s s_t
\end{array}\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= m_0+\overline{g}-\phi_d\overline{n}+A_1+B_1\overline{p}\left(1-\rho_p\right)+C_1\overline{n}\left(1-\rho_n\right) \\
&\quad +D_1\overline{q}\left(1-\rho_q\right)+G_1\overline{s}\left(1-\rho_s\right), \\
B_2 &= B_1\rho_p-g\left(B_1\sigma_{pp}+D_1\sigma_{qp}+\left(m_{\omega,p}+\gamma_g\sigma_{cp}\right)\right), \\
C_2 &= m_n+\phi_d+C_1\rho_n-g\left[C_1\sigma_{nn}+D_1\sigma_{qn}+\left(m_{\omega,n}-\left(\gamma_n+\gamma_g\right)\sigma_{cn}\right)\right], \\
D_2 &= D_1\rho_q+m_q, \\
G_2 &= G_1\rho_s-g\left(D_1\sigma_{qq}+G_1\sigma_{sq}+m_{\omega,q}\right).
\end{aligned}
$$

More generally,

$$
\begin{aligned}
A_n &= A_{n-1}+m_0+\overline{g}-\phi_d\overline{n}+B_{n-1}\overline{p}\left(1-\rho_p\right)+C_{n-1}\overline{n}\left(1-\rho_n\right) \\
&\quad +D_{n-1}\overline{q}\left(1-\rho_q\right)+G_{n-1}\overline{s}\left(1-\rho_s\right), \\
B_n &= B_{n-1}\rho_p-g\left(D_{n-1}\sigma_{qp}+B_{n-1}\sigma_{pp}+\left(m_{\omega,p}+\gamma_g\sigma_{cp}\right)\right), \\
C_n &= C_{n-1}\rho_n+m_n+\phi_d-g\left(C_{n-1}\sigma_{nn}+D_{n-1}\sigma_{qn}+\left(m_{\omega,n}-\left(\gamma_g+\gamma_n\right)\sigma_{cn}\right)\right), \\
D_n &= D_{n-1}\rho_q+m_q, \\
G_n &= G_{n-1}\rho_s-g\left(D_{n-1}\sigma_{qq}+G_{n-1}\sigma_{sq}+m_{\omega,q}\right).
\end{aligned}
$$

The price-dividend ratio can be expressed as

$$
\frac{P_t}{D_t}=\sum_{i=1}^{\infty}e^{A_i+B_i p_t+C_i n_t+D_i q_t+G_i s_t}.
$$

The first-order approximation of the log price-dividend ratio is

$$
pd_t=K_0^1+K_p^1 p_t+K_n^1 n_t+K_q^1 q_t+K_s^1 s_t
$$

where

$$K_p^1 = \frac{\sum_{i=1}^{\infty} B_i \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)}{\sum_{i=1}^{\infty} \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)},$$

$$K_n^1 = \frac{\sum_{i=1}^{\infty} C_i \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)}{\sum_{i=1}^{\infty} \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)},$$

$$K_q^1 = \frac{\sum_{i=1}^{\infty} D_i \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)}{\sum_{i=1}^{\infty} \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)},$$

$$K_s^1 = \frac{\sum_{i=1}^{\infty} G_i \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)}{\sum_{i=1}^{\infty} \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)},$$

and

$$K_0^1 = \ln\left(\sum_{i=1}^{\infty} \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)\right) - K_p^1\overline{p} - K_n^1\overline{n} - K_q^1\overline{q} - K_s^1\overline{s}.$$

The first-order approximation of $\ln\left(1 + \frac{P_{t+1}}{D_{t+1}}\right)$ is

$$\ln\left(1 + \frac{P_{t+1}}{D_{t+1}}\right) = K_0^2 + K_p^2 p_{t+1} + K_n^2 n_{t+1} + K_q^2 q_{t+1} + K_s^2 s_{t+1},$$

where

$$K_p^2 = \frac{\sum_{i=1}^{\infty} B_i \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)}{1 + \sum_{i=1}^{\infty} \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)},$$

$$K_n^2 = \frac{\sum_{i=1}^{\infty} C_i \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)}{1 + \sum_{i=1}^{\infty} \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)},$$

$$K_q^2 = \frac{\sum_{i=1}^{\infty} D_i \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)}{1 + \sum_{i=1}^{\infty} \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)},$$

$$K_s^2 = \frac{\sum_{i=1}^{\infty} G_i \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)}{1 + \sum_{i=1}^{\infty} \exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)},$$

and

$$K_0^2 = \ln\left(1+\sum_{i=1}^{\infty}\exp\left(A_i + B_i\overline{p} + C_i\overline{n} + D_i\overline{q} + G_i\overline{s}\right)\right) - K_p^2\overline{p} - K_n^2\overline{n} - K_q^2\overline{q} - K_s^2\overline{s}.$$

The expression for the log market return is therefore

$$\begin{aligned} r_{t+1} &= d_{t+1} + \ln\left(1+\frac{P_{t+1}}{D_{t+1}}\right) - pd_t \\ &= \overline{g} + \phi_d\left(n_t - \overline{n}\right) + \gamma_g\left(\sigma_{cp}\omega_{p,t+1} - \sigma_{cn}\omega_{n,t+1}\right) + \gamma_n\left(-\sigma_{cn}\omega_{n,t+1}\right) \\ &\quad + K_0^2 + K_p^2 p_{t+1} + K_n^2 n_{t+1} + K_q^2 q_{t+1} + K_s^2 s_{t+1} \\ &\quad - K_0^1 - K_p^1 p_t - K_n^1 n_t - K_q^1 q_t - K_s^1 s_t. \end{aligned}$$

which simplifies to

$$\begin{aligned} & r_{t+1} \\ = \; & \left\{ \begin{array}{c} \overline{g} - \phi_d\overline{n} + K_0^2 - K_0^1 + K_p^2\overline{p} - K_p^2\rho_p\overline{p} + K_n^2\overline{n} \\ -K_n^2\rho_n\overline{n} + K_q^2\overline{q} - K_q^2\rho_q\overline{q} + K_s^2\overline{s} - K_s^2\rho_s\overline{s} \end{array} \right\} \\ & + \left\{\gamma_g\sigma_{cp} + K_p^2\sigma_{pp} + K_q^2\sigma_{qp}\right\}\omega_{p,t+1} \\ & + \left\{K_q^2\sigma_{qn} - \left(\gamma_g + \gamma_n\right)\sigma_{cn} + K_n^2\sigma_{nn}\right\}\omega_{n,t+1} \\ & + \left\{K_p^2\rho_p - K_p^1\right\}p_t + \left\{K_n^2\rho_n - K_n^1 + \phi_d\right\}n_t + \left\{K_s^2\rho_s - K_s^1\right\}s_t \\ & + \left\{K_q^2\rho_q - K_q^1\right\}q_t + \left\{K_q^2\sigma_{qq} + K_s^2\sigma_{sq}\right\}\omega_{q,t+1}. \end{aligned}$$

This can be further simplified as

$$r_{t+1} = r_0 + r_p p_t + r_n n_t + r_q q_t + r_s s_t + r_{\omega p}\omega_{p,t+1} + r_{\omega n}\omega_{n,t+1} + r_{\omega q}\omega_{q,t+1},$$

where

$$\begin{aligned} r_0 &= \overline{g} - K_0^1 - \phi_d\overline{n} + K_0^2 + K_p^2\overline{p}\left(1-\rho_p\right) + K_n^2\overline{n}\left(1-\rho_n\right) \\ &\quad + K_q^2\overline{q}\left(1-\rho_q\right) + K_s^2\overline{s}\left(1-\rho_s\right), \\ r_p &= K_p^2\rho_p - K_p^1, \\ r_n &= K_n^2\rho_n - K_n^1 + \phi_d, \\ r_q &= K_q^2\rho_q - K_q^1, \\ r_s &= K_s^2\rho_s - K_s^1, \\ r_{\omega p} &= \gamma_g\sigma_{cp} + K_p^2\sigma_{pp} + K_q^2\sigma_{qp}, \\ r_{\omega n} &= K_q^2\sigma_{qn} - \left(\gamma_g + \gamma_n\right)\sigma_{cn} + K_n^2\sigma_{nn}, \\ r_{\omega q} &= K_q^2\sigma_{qq} + K_s^2\sigma_{sq}. \end{aligned}$$

Our goal now is to derive

$$
\begin{aligned}
\mathbb{E}_t\left[\left(R_{M,t\to t+1}-R_{f,t\to t+1}\right)^j\right] &= \sum_{k=0}^{j}\frac{j!\,(-1)^{j-k}}{(j-k)!k!}R_{f,t\to t+1}^{j-k}\mathbb{E}_t\left[R_{M,t\to t+1}^k\right],\\
\mathbb{E}_t\left[\left(R_{M,t\to t+1}-R_{f,t\to t+1}\right)^j\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] &= \sum_{k=0}^{j}\frac{j!\,(-1)^{j-k}}{(j-k)!k!}R_{f,t\to t+1}^{j-k}\mathbb{E}_t\left[R_{M,t\to t+1}^k\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right],\\
\mathbb{E}_t^*\left[\left(R_{M,t\to t+1}-R_{f,t\to t+1}\right)^j\right] &= \sum_{k=0}^{j}\frac{j!\,(-1)^{j-k}}{(j-k)!k!}R_{f,t\to t+1}^{j-k}\mathbb{E}_t^*\left[R_{M,t\to t+1}^k\right],\\
\mathbb{E}_t^*\left[\left(R_{M,t\to t+1}-R_{f,t\to t+1}\right)^j\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right] &= \sum_{k=0}^{j}\frac{j!\,(-1)^{j-k}}{(j-k)!k!}R_{f,t\to t+1}^{j-k}\mathbb{E}_t^*\left[R_{M,t\to t+1}^k\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right].
\end{aligned}
$$

We need to find

$$\mathbb{E}_t\left[R_{M,t\to t+1}^k\right],\ \mathbb{E}_t\left[R_{M,t\to t+1}^k\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right],\ \mathbb{E}_t^*\left[R_{M,t\to t+1}^k\right],\text{ and }\mathbb{E}_t^*\left[R_{M,t\to t+1}^k\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right].$$

Note that

$$
\begin{aligned}
\mathbb{E}_t\left[R_{M,t\to t+1}^k\right] &= \mathbb{E}_t\left[e^{kr_{t+1}}\right]\\
&= \mathbb{E}_t\left[e^{kr_0+kr_pp_t+kr_nn_t+kr_qq_t+kr_ss_t+kr_{\omega p}\omega_{p,t+1}+kr_{\omega n}\omega_{n,t+1}+kr_{\omega q}\omega_{q,t+1}}\right]\\
&= e^{kr_0+kr_pp_t+kr_nn_t+kr_qq_t+kr_ss_t}\mathbb{E}_t\left[e^{kr_{\omega p}\omega_{p,t+1}+kr_{\omega n}\omega_{n,t+1}+kr_{\omega q}\omega_{q,t+1}}\right]\\
&= \exp\left\{\begin{array}{c} kr_0+kr_pp_t+kr_nn_t+kr_qq_t+kr_ss_t\\ -p_tg\left(kr_{\omega p}\right)-n_tg\left(kr_{\omega n}\right)-s_tg\left(kr_{\omega q}\right)\end{array}\right\}.
\end{aligned}
$$

Next,

$$\mathbb{E}_t^*\left[R_{M,t\to t+1}^k\right]=R_{f,t+1}\mathbb{E}_t\left[M_{t\to t+1}R_{M,t\to t+1}^k\right]=R_{f,t+1}\mathbb{E}_t\left[e^{kr_{t+1}+m_{t+1}}\right].$$

Observe that

$$
\begin{aligned}
&kr_{t+1}+m_{t+1}\\
=&\left(\begin{array}{c} kr_0+kr_pp_t+kr_nn_t+kr_qq_t+kr_ss_t\\ +kr_{\omega p}\omega_{p,t+1}+kr_{\omega n}\omega_{n,t+1}+kr_{\omega q}\omega_{q,t+1}\\ +m_0+m_qq_t+m_nn_t+m_{\omega,p}\omega_{p,t+1}+m_{\omega,n}\omega_{n,t+1}+m_{\omega,q}\omega_{q,t+1}\end{array}\right)\\
=&\left(\begin{array}{c} kr_0+m_0+kr_pp_t+\left(kr_n+m_n\right)n_t\\ +\left(kr_q+m_q\right)q_t+kr_ss_t+\left(kr_{\omega p}+m_{\omega,p}\right)\omega_{p,t+1}\\ +\left(m_{\omega,n}+kr_{\omega n}\right)\omega_{n,t+1}+\left(kr_{\omega q}+m_{\omega,q}\right)\omega_{q,t+1}\end{array}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
& R_{f,t\to t+1}\mathbb{E}_t\left[e^{kr_{t+1}+m_{t+1}}\right] \\
= \quad & R_{f,t\to t+1}\mathbb{E}_t\left[\exp\left(\begin{array}{c} kr_0+m_0+kr_p p_t+(kr_n+m_n)\,n_t \\ +\,(kr_q+m_q q_t)\,q_t+kr_s s_t+(kr_{\omega p}+m_{\omega,p})\,\omega_{p,t+1} \\ +\,(m_{\omega,n}+kr_{\omega n})\,\omega_{n,t+1}+(kr_{\omega q}+m_{\omega,q})\,\omega_{q,t+1} \end{array}\right)\right].
\end{aligned}
$$

Hence

$$
\mathbb{E}_t^*\left[R_{M,t\to t+1}^k\right]=R_{f,t\to t+1}\exp\left(\begin{array}{c} kr_0+m_0+\{kr_p-g\,(kr_{\omega p}+m_{\omega,p})\}\,p_t \\ +\,\{kr_n+m_n-g\,(m_{\omega,n}+kr_{\omega n})\}\,n_t \\ +\,(kr_q+m_q)\,q_t+(kr_s-g\,(kr_{\omega q}+m_{\omega,q}))\,s_t \end{array}\right). \tag{IA.80}
$$

Now, let us find the truncated moments. Observe that

$$
\begin{aligned}
kr_{t+1} &= kr_0+kr_p p_t+kr_n n_t+kr_q q_t+kr_s s_t \\
&\quad +kr_{\omega p}\omega_{p,t+1}+kr_{\omega n}\omega_{n,t+1}+kr_{\omega q}\omega_{q,t+1} \\
&= kr_0+(kr_p-kr_{\omega p})\,p_t+(kr_n-kr_{\omega n})\,n_t \\
&\quad +kr_q q_t+(kr_s-kr_{\omega q})\,s_t \\
&\quad +kr_{\omega p}\,(\omega_{p,t+1}+p_t)+kr_{\omega n}\,(\omega_{n,t+1}+n_t) \\
&\quad +kr_{\omega q}\,(\omega_{q,t+1}+s_t) \\
&= \xi_{r,t}+Z_{t+1,r}
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_{r,t} &= kr_0+(kr_p-kr_{\omega p})\,p_t+(kr_n-kr_{\omega n})\,n_t+kr_q q_t+(kr_s-kr_{\omega q})\,s_t \\
Z_{t+1,r} &= kr_{\omega p}\,(\omega_{p,t+1}+p_t)+kr_{\omega n}\,(\omega_{n,t+1}+n_t)+kr_{\omega q}\,(\omega_{q,t+1}+s_t)\,.
\end{aligned}
$$

Hence

$$
\mathbb{E}_t\left[R_{M,t\to t+1}^k\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right]=e^{\xi_{r,t}}\mathbb{E}_t\left[e^{Z_{t+1,r}}\mathbb{I}_{\{Z_{t+1,r}>k\ln a-\xi_{r,t}\}}\right]
$$

Next, let us find $\mathbb{E}_t^*\left[R_{M,t\to t+1}^k\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right]$. Observe that

$$
\mathbb{E}_t^*\left[R_{M,t\to t+1}^k\mathbb{I}_{\{R_{M,t\to t+1}>a\}}\right]=R_{f,t\to t+1}\mathbb{E}_t\left[e^{m_{t+1}+\xi_{r,t}+Z_{t+1,r}}\mathbb{I}_{\{Z_{t+1,r}>k\log a-\xi_{r,t}\}}\right].
$$

Now, observe that

$$\begin{aligned}
& m_{t+1}+\xi_{r,t}+Z_{t+1,r} \\
= \; & m_0+m_q q_t+m_n n_t+m_{\omega,p}\omega_{p,t+1}+m_{\omega,n}\omega_{n,t+1}+m_{\omega,q}\omega_{q,t+1} \\
& +kr_0+\left(kr_p-kr_{\omega p}\right)p_t+\left(kr_n-kr_{\omega n}\right)n_t+kr_q q_t+\left(kr_s-kr_{\omega q}\right)s_t \\
& +kr_{\omega p}\left(\omega_{p,t+1}+p_t\right)+kr_{\omega n}\left(\omega_{n,t+1}+n_t\right)+kr_{\omega q}\left(\omega_{q,t+1}+s_t\right)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
& m_{t+1}+\xi_{r,t}+Z_{t+1,r} \\
= \; & m_0+kr_0 \\
& +\left(kr_p-kr_{\omega p}-m_{\omega,p}\right)p_t+\left(kr_n-kr_{\omega n}+m_n-m_{\omega,n}\right)n_t \\
& +\left(kr_s-kr_{\omega q}-m_{\omega,q}\right)s_t+\left(kr_q+m_q\right)q_t \\
& +\left(m_{\omega,p}+kr_{\omega p}\right)\left(\omega_{p,t+1}+p_t\right)+\left(m_{\omega,n}+kr_{\omega n}\right)\left(\omega_{n,t+1}+n_t\right) \\
& +\left(kr_{\omega q}+m_{\omega,q}\right)\left(\omega_{q,t+1}+s_t\right) \\
= \; & \zeta_r+Z_{t+1,m}
\end{aligned}$$

with

$$\begin{aligned}
\zeta_r \;=\; & m_0+kr_0+\left(kr_p-kr_{\omega p}-m_{\omega,p}\right)p_t+\left(kr_n-kr_{\omega n}+m_n-m_{\omega,n}\right)n_t \\
& +\left(kr_s-kr_{\omega q}-m_{\omega,q}\right)s_t+\left(kr_q+m_q\right)q_t
\end{aligned}$$

and

$$\begin{aligned}
Z_{t+1,m} \;=\; & \left(m_{\omega,p}+kr_{\omega p}\right)\left(\omega_{p,t+1}+p_t\right)+\left(m_{\omega,n}+kr_{\omega n}\right)\left(\omega_{n,t+1}+n_t\right) \\
& +\left(kr_{\omega q}+m_{\omega,q}\right)\left(\omega_{q,t+1}+s_t\right)
\end{aligned}$$

The model without preference shocks obtains when the state variable s_t is set to zero and the following parameter restrictions are imposed:[48]

$$\overline{s}=\rho_s=\sigma_{qq}=\phi_d=\phi_g=\gamma_n=m_n=m_{\omega q}=0. \tag{IA.81}$$

■

[48]Note that $\rho_p=\sigma_{pp}=0$ in both the model with and that without preference shocks.

IA.7.5 Disaster Risk Models

IA.7.5.1 Gabaix (2012)

We follow the Dew-Becker et al. (2017) implementation of Gabaix (2012) (including their parameter choices). The key processes that drive the economy are

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + \sigma_c \varepsilon_{c,t+1} + J_{c,t+1}, \\
L_{t+1} &= (1-\rho_L)\,\overline{L} + \rho_L L_t + \sigma_L \varepsilon_{L,t+1}, \\
\Delta d_{t+1} &= \eta \sigma_c \varepsilon_{c,t+1} - L_t \mathbb{I}_{J_{c_{t+1}} \neq 0},
\end{aligned}$$

where L_t represents the recovery rate of stocks in a disaster. $J_{c,t+1}$ is a disaster shock that affects the consumption and dividend processes, and follows a compound Poisson process given by

$$J_{c,t} = \sum_{i=1}^{N_t} \xi_{i,t} \text{ where } N_t \sim \text{Poisson}\,(\lambda) \text{ and } \xi_{i,t} \sim N\,(\mu_d, \sigma_d)\,.$$

The number of disasters, N_t, that occur each period follow a Poisson process with intensity λ. $\mathbb{I}$ is an indicator function. $J_{c,t+1}$, $\varepsilon_{L,t+1}$ and $\varepsilon_{c,t+1}$ are independent. In Gabaix (2012), the representative agent has a power utility utility preferences with risk aversion parameter γ. In this setting the SDF takes the form:

$$M_{t\to t+1} = \delta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

where δ is the time discount rate. The log-SDF is

$$m_{t+1} = \log \delta - \gamma \Delta c_{t+1}.$$

Thus, the log of the return on the risk-free asset is

$$\begin{aligned}
\log R_{f,t+1} &= -\log\left(\mathbb{E}_t e^{m_{t+1}}\right) \\
&= -\log\left(\mathbb{E}_t e^{(\log\delta - \gamma\Delta c_{t+1})}\right) \\
&= -\log\left(\mathbb{E}_t \exp\left\{\log\delta - \gamma\mu_c - \gamma\sigma_c\varepsilon_{c,t+1} - \gamma J_{c,t+1}\right\}\right) \\
&= -\log\delta + \gamma\mu_c - \log \mathbb{E}_t \exp\left\{-\gamma\sigma_c\varepsilon_{c,t+1} - \gamma J_{c,t+1}\right\}.
\end{aligned}$$

Note that

$$\mathbb{E}_t \exp\left\{-\gamma\sigma_c\varepsilon_{c,t+1} - \gamma J_{c,t+1}\right\} = \left(\mathbb{E}_t \exp\left\{-\gamma\sigma_c\varepsilon_{c,t+1}\right\}\right)\left(\mathbb{E}_t \exp\left\{-\gamma J_{c,t+1}\right\}\right).$$

Since

$$\mathbb{E}_t \exp\{-\gamma J_{c,t+1}\} = \exp\left(\lambda\left(e^{\varphi_G} - 1\right)\right)$$

with

$$\varphi_G = -\gamma\mu_d + \frac{1}{2}\gamma^2\sigma_d^2.$$

Hence

$$\mathbb{E}_t \exp\{-\gamma\sigma_c\varepsilon_{c,t+1} - \gamma J_{c,t+1}\} = \exp\left\{\frac{1}{2}\gamma^2\sigma_c^2 + \lambda\left(e^{\varphi_G} - 1\right)\right\}.$$

Thus

$$\log R_{f,t+1} = -\log\delta + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c^2 - \lambda\left(e^{\varphi_G} - 1\right).$$

Now, let us compute the physical and risk neutral moments. We use the Campbell and Shiller (1988) to approximate multiples of the log market return as

$$nr_{m,t+1} \approx n\kappa_0 + n\kappa_1 pd_{t+1} - npd_t + n\Delta d_{t+1}$$

Next, we project the log price-dividend ratio (pd_t) on the stock recovery rate, L_t, at each date. In this model, the log price-dividend ratio is not linear in the state variable, L_t. To approximate pd_t as a function of L_t, we project it onto basis functions and impose that the Euler equation holds (approximately) at each point in L_t. Denote the projected price-dividend ratio using

$$pd_t = f\left[L_t\right].$$

The physical non-central moment of the market return is

$$\mathbb{E}_t\left[R^n_{M,t\to t+1}\right] = \mathbb{E}_t\left[e^{nr_{m,t+1}}\right]$$

Similarly, the non-central truncated moment

$$\mathbb{E}_t\left[R^n_{M,t\to t+1}\mathbb{I}_{R_{M,t\to t+1}>a}\right] = \mathbb{E}_t\left[e^{nr_{m,t+1}}\mathbb{I}_{R_{M,t\to t+1}>a}\right]$$

Expressions for the non-central risk neutral and truncated non-central risk neutral moments are

$$\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\right] = R_{f,t\to t+1}\mathbb{E}_t\left[e^{nr_{m,t+1}+m_{t+1}}\right],$$

and

$$\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\mathbb{I}_{R_{M,t\to t+1}>a}\right] = R_{f,t\to t+1}\mathbb{E}_t\left[e^{nr_{m,t+1}+m_{t+1}}\mathbb{I}_{R_{M,t\to t+1}>a}\right].$$

Multiples of the log market return are given by

$$nr_{m,t+1} \approx n\kappa_0 + n\kappa_1 f\left[L_{t+1}\right] - nf\left[L_t\right] + n\Delta d_{t+1}.$$

Given the state variable, L_t, we can use this expression to compute the physical and risk-neutral moments above.

IA.7.5.2 Wachter (2013)

We follow the Dew-Becker et al. (2017) discretization of Wachter (2013) (including their parameter choices) so that we can evaluate a monthly-frequency version of this model. The key processes that drive the economy are given by

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + \sigma_c z_{c,t+1} + J_{t+1}, \\
\lambda_{t+1} &= (1-\rho_\lambda)\,\mu_\lambda + \rho_\lambda \lambda_t + \sigma_\lambda \sqrt{\lambda_t} z_{\lambda,t+1}, \\
\Delta d_{t+1} &= \phi\mu_c + \phi\sigma_c z_{c,t+1} + \phi J_{t+1},
\end{aligned}$$

where the shocks $z_{c,t+1}$ and $z_{\lambda,t+1}$ are uncorrelated and follow standard normal distributions. J_{t+1} follows a compound Poisson process given by

$$J_{t+1} = \sum_{i=1}^{N_{t+1}} \xi_{i,t+1} \text{ where } N_{t+1} \sim \text{Poisson}\,(\lambda_t) \text{ and } \xi_{i,t+1} \sim N\,(\mu_d, \sigma_d)\,.$$

The number of disasters, N_t, that occur each period follow a Poisson process with intensity λ_t. The variables $z_{c,t+1}$, $z_{\lambda,t+1}$, and J_{t+1} are assumed to be independent. Following Dew-Becker et al. (2017), the household utility is

$$\upsilon_t = (1-\beta)\log C_t + \frac{\beta}{1-\alpha}\log \mathbb{E}_t\left(e^{((1-\alpha)\upsilon_{t+1})}\right), \tag{IA.82}$$

and the log-SDF takes the form

$$m_{t+1} = \log\beta - \Delta c_{t+1} + (1-\alpha)\,\upsilon_{t+1} - \log\left(\mathbb{E}_t \exp\left\{(1-\alpha)\,\upsilon_{t+1}\right\}\right).$$

Main Results

Result IA.13. *Given the state variable, λ_t, the model-implied log price-dividend ratio is given by*

$$\log\frac{P_t}{D_t} = A_{0,m} + A_{1,m}\lambda_t,$$

where $A_{0,m}$ and $A_{1,m}$ are given below.

Proof. See below. ■

Result IA.14. *For any $n > 0$, the non-central physical moment of the market return is*

$$\mathbb{E}_t\left(R^n_{M,t+1}\right) = \exp\left\{\begin{array}{c} n\kappa_0 + n\left(\kappa_1 - 1\right) A_{0,m} + n\kappa_1 A_{1,m}\left(1-\rho_\lambda\right)\mu_\lambda \\ \\ +n\phi\mu_c + \frac{1}{2}\left(n\phi\sigma_c\right)^2 \\ \\ +\left\{n\left(\kappa_1\rho_\lambda - 1\right) A_{1,m} + \frac{1}{2}\left(n\kappa_1 A_{1,m}\sigma_\lambda\right)^2 + \left(e^{\varphi_\phi} - 1\right)\right\}\lambda_t \end{array}\right\}$$

with

$$\varphi_\phi = \left(n\phi\right)\mu_d + \frac{1}{2}\left(n\phi\right)^2\sigma_d^2.$$

The truncated non-central physical moment is

$$\mathbb{E}_t\left(R^n_{M,t\to t+1}1_{R_{M,t\to t+1}>a}\right) = \mathbb{E}_t\left(e^{\mathcal{A}_t + Z_{t+1}}1_{\mathcal{A}_t + Z_{t+1} > n\log a}\right),$$

where

$$\begin{aligned} \mathcal{A}_t &= n\kappa_0 + n\left(\kappa_1 - 1\right)A_{0,m} + n\kappa_1 A_{1,m}\left(1-\rho_\lambda\right)\mu_\lambda \\ &\quad + n\left(\kappa_1\rho_\lambda - 1\right)A_{1,m}\lambda_t + n\phi\mu_c \\ Z_{t+1} &= n\kappa_1 A_{1,m}\sigma_\lambda\sqrt{\lambda_t}z_{\lambda,t+1} + n\phi\sigma_c z_{c,t+1} + n\phi J_{t+1}. \end{aligned}$$

All parameters are defined below.

Proof. See below. ■

Result IA.15. *The non-central risk neutral-moments of the market return are*

$$\mathbb{E}^*_t\left[R^n_{M,t\to t+1}\right] = R_{f,t\to t+1}\exp\left(\mathbb{A}^*_0 + \mathbb{A}^*_1\lambda_t\right)$$

where

$$\mathbb{A}^*_0 = \left\{\begin{array}{c} n\kappa_0 + n\left(\kappa_1 - 1\right)A_{0,m} + n\kappa_1 A_{1,m}\left(1-\rho_\lambda\right)\mu_\lambda \\ \\ +A^s_0 + A^s_2\left(1-\rho_\lambda\right)\mu_\lambda + n\phi\mu_c + A^s_3\mu_c \\ \\ +\frac{1}{2}\left(n\phi + A^s_3\right)^2\sigma_c^2 \end{array}\right\}$$

and

$$\mathbb{A}^*_1 = \left\{\begin{array}{c} n\left(\kappa_1\rho_\lambda - 1\right)A_{1,m} + A^s_1 + A^s_2\rho_\lambda \\ \\ +\frac{1}{2}\left(n\kappa_1 A_{1,m} + A^s_2\right)^2\sigma_\lambda^2 + \left(e^{\varphi^*_{\phi,s}} - 1\right) \end{array}\right\}$$

with

$$\varphi_{\phi,s}^{*}=\left(n\phi+A_{3}^{s}\right)\mu_{d}+\frac{1}{2}\left(n\phi+A_{3}^{s}\right)^{2}\sigma_{d}^{2}.$$

The truncated non-central risk-neutral moments of the market return are

$$\mathbb{E}_{t}^{*}\left(R_{M,t+1}^{n}1_{R_{M,t+1}>a}\right)=R_{f,t+1}\mathbb{E}_{t}\left\{\mathcal{N}\left[d_{1,t+1}^{*}\right]\left(\exp\left\{\mathcal{A}_{t}^{*}\right\}\right)\exp\left\{\mu_{z,t+1}^{*}+\frac{1}{2}\sigma_{z,t+1}^{*2}\right\}\right\}$$

where

$$\mathbb{E}_{t}^{*}\left(R_{M,t+1}^{n}1_{R_{M,t+1}>a}\right)=R_{f,t+1}\mathbb{E}_{t}\left(e^{m_{t+1}+nr_{m,t+1}}1_{R_{M,t+1}>a}\right)$$

Note that

$$m_{t+1}+nr_{m,t+1}=Z_{t+1}^{*}+\mathcal{A}_{t}^{*}$$

where

$$Z_{t+1}^{*}=\left\{n\kappa_{1}A_{1,m}+A_{2}^{s}\right\}\sigma_{\lambda}\sqrt{\lambda_{t}}z_{\lambda,t+1}+\left(n\phi+A_{3}^{s}\right)\sigma_{c}z_{c,t+1}+\left(n\phi+A_{3}^{s}\right)J_{t+1}$$

and

$$\begin{aligned}\mathcal{A}_{t}^{*} &= \left\{\begin{array}{c} n\kappa_{0}+n\left(\kappa_{1}-1\right)A_{0,m}+n.\kappa_{1}A_{1,m}\left(1-\rho_{\lambda}\right)\mu_{\lambda} \\ +A_{0}^{s}+A_{2}^{s}\left(1-\rho_{\lambda}\right)\mu_{\lambda}+n\phi\mu_{c}+A_{3}^{s}\mu_{c}\end{array}\right\} \\ &\quad +\left\{n\left(\kappa_{1}\rho_{\lambda}-1\right)A_{1,m}+A_{1}^{s}+A_{2}^{s}\rho_{\lambda}\right\}\lambda_{t}.\end{aligned}$$

All parameters are defined below.

Proof. See below. ■

Derivations and Proofs We start with a conjecture that the log utility-consumption ratio is a linear function of the intensity λ_t

$$\upsilon_{t}-\log C_{t}=A_{0}+A_{1}\lambda_{t}. \tag{IA.83}$$

This allows us to express $(1-\alpha)\,\upsilon_{t+1}$ as

$$\begin{aligned}\left(1-\alpha\right)\upsilon_{t+1} &= \left(1-\alpha\right)\log C_{t}+\left(1-\alpha\right)\left(\log C_{t+1}-\log C_{t}\right) \\ &\quad +\left(1-\alpha\right)A_{0}+A_{1}\left(1-\alpha\right)\lambda_{t+1}.\end{aligned}$$

Hence

$$
\begin{aligned}
\mathbb{E}_t \exp\left\{(1-\alpha)\, v_{t+1}\right\} &= \left(\exp\left\{(1-\alpha)\log C_t\right\}\right)\mathbb{E}_t\left(\exp\left\{\begin{array}{c}(1-\alpha)\,\Delta c_{t+1}+(1-\alpha)\,A_0\\ +A_1\,(1-\alpha)\,\lambda_{t+1}\end{array}\right\}\right)\\
&= \left(\exp\left\{\begin{array}{c}(1-\alpha)\log C_t+A_1\,(1-\alpha)\,\rho_\lambda\lambda_t\\ (1-\alpha)\,\mu_c+(1-\alpha)\,A_0\\ +A_1\,(1-\alpha)\,(1-\rho_\lambda)\,\mu_\lambda\end{array}\right\}\right)\\
&\quad \times\mathbb{E}_t\left(\exp\left\{\begin{array}{c}(1-\alpha)\,\sigma_c z_{c,t+1}+(1-\alpha)\,J_{t+1}\\ +A_1\,(1-\alpha)\,\sigma_\lambda\sqrt{\lambda_t}z_{\lambda,t+1}\end{array}\right\}\right)
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\mathbb{E}_t \exp\left\{(1-\alpha)\, v_{t+1}\right\} &= \left(\exp\left\{\begin{array}{c}(1-\alpha)\log C_t+A_1\,(1-\alpha)\,\rho_\lambda\lambda_t\\ (1-\alpha)\,\mu_c+(1-\alpha)\,A_0\\ +A_1\,(1-\alpha)\,(1-\rho_\lambda)\,\mu_\lambda\\ +\frac{1}{2}\,(1-\alpha)^2\,\sigma_c^2+\frac{1}{2}A_1^2\,(1-\alpha)^2\,\sigma_\lambda^2\lambda_t\end{array}\right\}\right)\\
&\quad \times\mathbb{E}_t\left(\exp\left\{(1-\alpha)\,J_{t+1}\right\}\right).
\end{aligned}
$$

Note that

$$
\mathbb{E}_t\left(\exp\left\{(1-\alpha)\,J_{t+1}\right\}\right)=\mathbb{E}_t\left(\mathbb{E}_t\left(\exp\left\{(1-\alpha)\,J_{t+1}\right\}|N_{t+1}\right)\right).
$$

Thus

$$
\begin{aligned}
\mathbb{E}_t\left(\exp\left\{(1-\alpha)\,J_{t+1}\right\}|N_{t+1}\right) &= \exp\left\{(1-\alpha)\,\mu_d N_{t+1}+\frac{N_{t+1}}{2}\,(1-\alpha)^2\,\sigma_d^2\right\}\\
&= \exp\left\{\varphi_\alpha N_{t+1}\right\},
\end{aligned}
$$

where

$$
\varphi_\alpha=(1-\alpha)\,\mu_d+\frac{1}{2}\,(1-\alpha)^2\,\sigma_d^2.
$$

Therefore,

$$\mathbb{E}_t\left(\exp\left\{(1-\alpha)J_{t+1}\right\}\right) = \mathbb{E}_t\left(\exp\left\{\varphi_\alpha N_{t+1}\right\}\right) = \exp\left(\lambda_t\left(e^{\varphi_\alpha}-1\right)\right).$$

Consequently,

$$\mathbb{E}_t \exp\left\{(1-\alpha)\upsilon_{t+1}\right\} = \exp\left\{\begin{array}{c}(1-\alpha)\log C_t + A_1(1-\alpha)\rho_\lambda\lambda_t \\ (1-\alpha)\mu_c + (1-\alpha)A_0 \\ +A_1(1-\alpha)(1-\rho_\lambda)\mu_\lambda \\ +\frac{1}{2}(1-\alpha)^2\sigma_c^2 + \frac{1}{2}A_1^2(1-\alpha)^2\sigma_\lambda^2\lambda_t \\ +\lambda_t\left(e^{\varphi_\alpha}-1\right)\end{array}\right\}.$$

Note that:

$$\upsilon_t = (1-\beta)\log C_t + \frac{\beta}{1-\alpha}\log \mathbb{E}_t\left(e^{((1-\alpha)\upsilon_{t+1})}\right).$$

We then replace this expression in Equation IA.82 and show:

$$\upsilon_t = (1-\beta)\log C_t + \left(\frac{\beta}{1-\alpha}\right)\left\{\begin{array}{c}(1-\alpha)\log C_t + A_1(1-\alpha)\rho_\lambda\lambda_t \\ (1-\alpha)\mu_c + (1-\alpha)A_0 \\ +A_1(1-\alpha)(1-\rho_\lambda)\mu_\lambda \\ +\frac{1}{2}(1-\alpha)^2\sigma_c^2 + \frac{1}{2}A_1^2(1-\alpha)^2\sigma_\lambda^2\lambda_t \\ +\lambda_t\left(e^{\varphi_\alpha}-1\right)\end{array}\right\}$$

which simplifies to

$$\begin{aligned}\upsilon_t - \log C_t \quad = \quad & \left\{\beta\mu_c + \beta A_0 + A_1\beta(1-\rho_\lambda)\mu_\lambda + \frac{1}{2}\beta(1-\alpha)\sigma_c^2\right\} \\ & + \left\{A_1\beta\rho_\lambda + \frac{1}{2}A_1^2\beta(1-\alpha)\sigma_\lambda^2 + \left(\frac{\beta}{1-\alpha}\right)\left(e^{\varphi_\alpha}-1\right)\right\}\lambda_t.\end{aligned}$$

By identification with Equation IA.83, we deduce

$$\begin{aligned} A_0 &= \beta\mu_c + \beta A_0 + A_1\beta\left(1-\rho_\lambda\right)\mu_\lambda + \frac{1}{2}\beta\left(1-\alpha\right)\sigma_c^2 \\ A_1 &= A_1\beta\rho_\lambda + \frac{1}{2}A_1^2\beta\left(1-\alpha\right)\sigma_\lambda^2 + \left(\frac{\beta}{1-\alpha}\right)\left(e^{\varphi_\alpha}-1\right). \end{aligned}$$

Thus,

$$A_0 = \frac{1}{1-\beta}\left\{\beta\mu_c + A_1\beta\left(1-\rho_\lambda\right)\mu_\lambda + \frac{1}{2}\beta\left(1-\alpha\right)\sigma_c^2\right\}$$

and

$$0 = 2A_1\left(\beta\rho_\lambda - 1\right) + A_1^2\beta\left(1-\alpha\right)\sigma_\lambda^2 + 2\left(\frac{\beta}{1-\alpha}\right)\left(e^{\varphi_\alpha}-1\right)$$

which implies

$$A_1 = \frac{\left(1-\beta\rho_\lambda\right) \pm \sqrt{\left(\beta\rho_\lambda - 1\right)^2 - \left\{2\left(\frac{\beta}{1-\alpha}\right)\left(e^{\varphi_\alpha}-1\right)\right\}\left\{\beta\left(1-\alpha\right)\sigma_\lambda^2\right\}}}{\beta\left(1-\alpha\right)\sigma_\lambda^2}.$$

We choose the negative A_1. Note that

$$\upsilon_t - \log C_t = A_0 + A_1\lambda_t.$$

The log-SDF takes the form

$$m_{t+1} = \log\beta - \Delta c_{t+1} + \left(1-\alpha\right)\upsilon_{t+1} - \log\left(\mathbb{E}_t \exp\left\{\left(1-\alpha\right)\upsilon_{t+1}\right\}\right)$$

which simplifies to

$$\begin{aligned} m_{t+1} &= \log\beta - \Delta c_{t+1} + \left(1-\alpha\right)A_0 + \left(1-\alpha\right)A_1\lambda_{t+1} \\ &\quad + \left(1-\alpha\right)\log C_{t+1} - \log\left(\mathbb{E}_t \exp\left\{\left(1-\alpha\right)\upsilon_{t+1}\right\}\right) \end{aligned}$$

and hence

$$\begin{aligned} m_{t+1} &= \left\{\begin{array}{c} \log\beta - \left(1-\alpha\right)\mu_c \\ \\ -A_1\left(1-\alpha\right)\left(1-\rho_\lambda\right)\mu_\lambda - \frac{1}{2}\left(1-\alpha\right)^2\sigma_c^2 \end{array}\right\} \\ &\quad -\alpha\Delta c_{t+1} + \left(1-\alpha\right)A_1\lambda_{t+1} \\ &\quad + \left\{-A_1\left(1-\alpha\right)\rho_\lambda - \frac{1}{2}A_1^2\left(1-\alpha\right)^2\sigma_\lambda^2 - \left(e^{\varphi_\alpha}-1\right)\right\}\lambda_t. \end{aligned}$$

Thus,

$$m_{t+1} = A_0^{\mathrm{s}} + A_1^{\mathrm{s}} \lambda_t + A_2^{\mathrm{s}} \lambda_{t+1} + A_3^{\mathrm{s}} \Delta c_{t+1},$$

where

$$\begin{aligned}
A_0^{\mathrm{s}} &= \log \beta - (1-\alpha)\mu_c - A_1 (1-\alpha)(1-\rho_\lambda)\mu_\lambda - \frac{1}{2}(1-\alpha)^2 \sigma_c^2, \\
A_1^{\mathrm{s}} &= -A_1 (1-\alpha)\rho_\lambda - \frac{1}{2} A_1^2 (1-\alpha)^2 \sigma_\lambda^2 - \left(e^{\varphi_\alpha} - 1\right), \\
A_2^{\mathrm{s}} &= A_1 (1-\alpha), \\
A_3^{\mathrm{s}} &= -\alpha.
\end{aligned}$$

The return on the risk-free asset is given by

$$\log R_{f,t\to t+1} = -\log \mathbb{E}_t \left(M_{t\to t+1}\right).$$

Note that

$$\begin{aligned}
& \mathbb{E}_t \left(M_{t\to t+1}\right) \\
= \ & \mathbb{E}_t \exp\left\{A_0^{\mathrm{s}} + A_1^{\mathrm{s}} \lambda_t + A_2^{\mathrm{s}} \lambda_{t+1} + A_3^{\mathrm{s}} \Delta c_{t+1}\right\} \\
= \ & \left(\exp\left\{A_0^{\mathrm{s}} + A_1^{\mathrm{s}} \lambda_t\right\}\right) \mathbb{E}_t \exp\left\{A_2^{\mathrm{s}} \lambda_{t+1} + A_3^{\mathrm{s}} \Delta c_{t+1}\right\} \\
= \ & \left(\exp\left\{\begin{array}{c} A_0^{\mathrm{s}} + A_3^{\mathrm{s}}\mu_c + A_2^{\mathrm{s}}(1-\rho_\lambda)\mu_\lambda + \frac{1}{2}\left(A_3^{\mathrm{s}}\right)^2 \sigma_c^2 \\ +A_1^{\mathrm{s}}\lambda_t + A_2^{\mathrm{s}}\rho_\lambda\lambda_t + \frac{1}{2}\left(A_2^{\mathrm{s}}\right)^2 \sigma_\lambda^2 \lambda_t \end{array}\right\}\right) \mathbb{E}_t \exp\left\{A_3^{\mathrm{s}} J_{t+1}\right\}.
\end{aligned}$$

Now, note that

$$\mathbb{E}_t \left(\exp\left\{A_3^{\mathrm{s}} J_{t+1}\right\}\right) = \mathbb{E}_t \exp\left\{A_3^{\mathrm{s}}\mu_d N_{t+1} + \frac{N_{t+1}}{2}\left(A_3^{\mathrm{s}}\right)^2 \sigma_d^2\right\} = \mathbb{E}_t \exp\left\{\varphi^s N_{t+1}\right\},$$

where

$$\varphi_s = A_3^{\mathrm{s}}\mu_d + \frac{1}{2}\left(A_3^{\mathrm{s}}\right)^2 \sigma_d^2.$$

Hence,

$$\mathbb{E}_t \left(\exp\left\{A_3^{\mathrm{s}} J_{t+1}\right\}\right) = \mathbb{E}_t \left\{\exp\left(\lambda_t \left(e^{\varphi_s} - 1\right)\right)\right\}.$$

Finally,

$$\mathbb{E}_t\left(M_{t\to t+1}\right) = \exp\left\{\begin{array}{c} A_0^{\mathrm{s}} + A_3^{\mathrm{s}}\mu_c + A_2^{\mathrm{s}}\left(1-\rho_\lambda\right)\mu_\lambda + \frac{1}{2}\left(A_3^{\mathrm{s}}\right)^2\sigma_c^2 \\ +A_1^{\mathrm{s}}\lambda_t + A_2^{\mathrm{s}}\rho_\lambda\lambda_t + \frac{1}{2}\left(A_2^{\mathrm{s}}\right)^2\sigma_\lambda^2\lambda_t \\ +\lambda_t\left(e^{\varphi_s}-1\right) \end{array}\right\}.$$

and

$$\log R_{f,t\to t+1} = A_0^{\mathrm{rf}} + A_1^{\mathrm{rf}}\lambda_t,$$

where

$$\begin{aligned} A_0^{\mathrm{rf}} &= -\left\{ A_0^{\mathrm{s}} + A_3^{\mathrm{s}}\mu_c + A_2^{\mathrm{s}}\left(1-\rho_\lambda\right)\mu_\lambda + \tfrac{1}{2}\left(A_3^{\mathrm{s}}\right)^2\sigma_c^2 \right\}, \\ A_1^{\mathrm{rf}} &= -\left\{A_1^{\mathrm{s}} + A_2^{\mathrm{s}}\rho_\lambda + \frac{1}{2}\left(A_2^{\mathrm{s}}\right)^2\sigma_\lambda^2 + \left(e^{\varphi_s}-1\right)\right\}. \end{aligned}$$

Following Dew-Becker et al. (2017), we use the Campbell and Shiller (1988) approximation to express the log market return as

$$r_{m,t+1} \approx \kappa_0 + \kappa_1 pd_{t+1} - pd_t + \Delta d_{t+1}$$

where pd_t is the log price-dividend ratio.

We provide a proof of Result IA.13 below.

Proof. Proof of Result IA.13. We conjecture that pd_t is linear in the state variables

$$pd_t = A_{0,m} + A_{1,m}\lambda_t. \tag{IA.84}$$

Thus,

$$\begin{aligned} r_{m,t+1} &\approx \kappa_0 + \kappa_1\left(A_{0,m} + A_{1,m}\lambda_{t+1}\right) - \left(A_{0,m} + A_{1,m}\lambda_t\right) \\ &\quad + \left(\phi\mu_c + \phi\sigma_c z_{c,t+1} + \phi J_{t+1}\right) \\ &= \kappa_0 + \kappa_1 A_{0,m} + \kappa_1 A_{1,m}\lambda_{t+1} - A_{0,m} - A_{1,m}\lambda_t \\ &\quad +\phi\mu_c + \phi\sigma_c z_{c,t+1} + \phi J_{t+1}. \end{aligned}$$

This expands to

$$\begin{aligned} r_{m,t+1} &= \kappa_0 + \left(\kappa_1 - 1\right)A_{0,m} + \kappa_1 A_{1,m}\left(1-\rho_\lambda\right)\mu_\lambda + \left(\kappa_1\rho_\lambda - 1\right)A_{1,m}\lambda_t \\ &\quad +\kappa_1 A_{1,m}\sigma_\lambda\sqrt{\lambda_t} z_{\lambda,t+1} + \phi\mu_c + \phi\sigma_c z_{c,t+1} + \phi J_{t+1}. \end{aligned}$$

To identify $A_{0,m}$ and $A_{1,m}$, we use the Euler equation

$$\log\left(\mathbb{E}_t e^{(m_{t+1}+r_{m,t+1})}\right) = 0.$$

Note that

$$\begin{aligned} m_{t+1}+r_{m,t+1} &= \{\kappa_0 + (\kappa_1 - 1) A_{0,m} + \kappa_1 A_{1,m} (1-\rho_\lambda) \mu_\lambda\} + \phi\mu_c + A_0^{\mathrm{s}} \\ &\quad + \{(\kappa_1\rho_\lambda - 1) A_{1,m} + A_1^{\mathrm{s}}\} \lambda_t + \kappa_1 A_{1,m}\sigma_\lambda \sqrt{\lambda_t} z_{\lambda,t+1} \\ &\quad + \phi\sigma_c z_{c,t+1} + \phi J_{t+1} + A_2^{\mathrm{s}}\lambda_{t+1} + A_3^{\mathrm{s}}\Delta c_{t+1} \end{aligned}$$

which simplifies to

$$\begin{aligned} m_{t+1}+r_{m,t+1} &= \left\{ \begin{array}{c} \kappa_0 + (\kappa_1 - 1) A_{0,m} + \kappa_1 A_{1,m} (1-\rho_\lambda) \mu_\lambda \\ \\ +A_3^{\mathrm{s}}\mu_c + \phi\mu_c + A_0^{\mathrm{s}} + A_2^{\mathrm{s}} (1-\rho_\lambda) \mu_\lambda \end{array} \right\} \\ &\quad + \{(\kappa_1\rho_\lambda - 1) A_{1,m} + A_1^{\mathrm{s}} + A_2^{\mathrm{s}}\rho_\lambda\} \lambda_t \\ &\quad + (\kappa_1 A_{1,m} + A_2^{\mathrm{s}}) \sigma_\lambda \sqrt{\lambda_t} z_{\lambda,t+1} \\ &\quad + (\phi\sigma_c + A_3^{\mathrm{s}}\sigma_c) z_{c,t+1} + (\phi + A_3^{\mathrm{s}}) J_{t+1} \end{aligned}$$

Thus

$$\begin{aligned} & \log \mathbb{E}_t e^{(m_{t+1}+r_{m,t+1})} \\ = & \left\{ \begin{array}{c} \kappa_0 + (\kappa_1 - 1) A_{0,m} + \kappa_1 A_{1,m} (1-\rho_\lambda) \mu_\lambda \\ \\ + (A_3^{\mathrm{s}} + \phi) \mu_c + A_0^{\mathrm{s}} + A_2^{\mathrm{s}} (1-\rho_\lambda) \mu_\lambda + \frac{1}{2} (\phi\sigma_c + A_3^{\mathrm{s}}\sigma_c)^2 \end{array} \right\} \\ & + \left\{ \begin{array}{c} (\kappa_1\rho_\lambda - 1) A_{1,m} + A_1^{\mathrm{s}} + A_2^{\mathrm{s}}\rho_\lambda \\ \\ +\frac{1}{2} (\kappa_1 A_{1,m} + A_2^{\mathrm{s}})^2 \sigma_\lambda^2 \end{array} \right\} \lambda_t + \log \mathbb{E}_t \left(\exp\left((\phi + A_3^{\mathrm{s}}) J_{t+1}\right)\right). \end{aligned}$$

Note that

$$\mathbb{E}_t \left(\exp\left((\phi + A_3^{\mathrm{s}}) J_{t+1}\right)\right) = \mathbb{E}_t \exp\left(\varphi_{\phi,s} N_{t+1}\right) = \exp\left(\lambda_t \left(e^{\varphi_{\phi,s}} - 1\right)\right),$$

where

$$\varphi_{\phi,s} = (\phi + A_3^{\mathrm{s}}) \mu_d + \frac{1}{2} (\phi + A_3^{\mathrm{s}})^2 \sigma_d^2.$$

Therefore,

$$\begin{aligned}
& \log \mathbb{E}_t e^{(m_{t+1}+r_{m,t+1})} \\
= & \left\{ \begin{array}{c} \kappa_0 + (\kappa_1 - 1) A_{0,m} + \kappa_1 A_{1,m} (1-\rho_\lambda) \mu_\lambda \\ + (A_3^{\mathrm{s}} + \phi) \mu_c + A_0^{\mathrm{s}} + A_2^{\mathrm{s}} (1-\rho_\lambda) \mu_\lambda + \frac{1}{2} (\phi\sigma_c + A_3^{\mathrm{s}} \sigma_c)^2 \end{array} \right\} \\
& + \left\{ \begin{array}{c} (\kappa_1 \rho_\lambda - 1) A_{1,m} + A_1^{\mathrm{s}} + A_2^{\mathrm{s}} \rho_\lambda \\ + \frac{1}{2} (\kappa_1 A_{1,m} + A_2^{\mathrm{s}})^2 \sigma_\lambda^2 + (e^{\varphi_{\phi,s}} - 1) \end{array} \right\} \lambda_t. \qquad \text{(IA.85)}
\end{aligned}$$

Since, $\log \mathbb{E}_t e^{(m_{t+1}+r_{m,t+1})} = 0$, this implies that each term of Equation IA.85 is zero. As a result, the coefficients from the conjecture in Equation IA.84 are

$$A_{0,m} = \frac{1}{1-\kappa_1} \left\{ \begin{array}{c} \kappa_0 + \kappa_1 A_{1,m} (1-\rho_\lambda) \mu_\lambda + (A_3^{\mathrm{s}} + \phi) \mu_c \\ + A_0^{\mathrm{s}} + A_2^{\mathrm{s}} (1-\rho_\lambda) \mu_\lambda + \frac{1}{2} (\phi\sigma_c + A_3^{\mathrm{s}} \sigma_c)^2 \end{array} \right\}$$

and

$$2\left\{(\kappa_1\rho_\lambda - 1) + \kappa_1 A_2^{\mathrm{s}} \sigma_\lambda^2\right\} A_{1,m} + \kappa_1^2 \sigma_\lambda^2 A_{1,m}^2 + 2\left\{\frac{1}{2}(A_2^{\mathrm{s}})^2 \sigma_\lambda^2 + A_1^{\mathrm{s}} + A_2^{\mathrm{s}} \rho_\lambda + (e^{\varphi_{\phi,s}} - 1)\right\}$$

which simplifies to

$$\mathcal{C} + 2\mathcal{B}.A_{1,m} + \mathcal{A}.A_{1,m}^2 = 0$$

where

$$\begin{aligned}
\mathcal{C} &= 2\left\{A_1^{\mathrm{s}} + A_2^{\mathrm{s}} \rho_\lambda + (e^{\varphi_{\phi,s}} - 1) + \frac{1}{2}(A_2^{\mathrm{s}})^2 \sigma_\lambda^2\right\} \\
\mathcal{B} &= \left\{(\kappa_1\rho_\lambda - 1) + \kappa_1 A_2^{\mathrm{s}} \sigma_\lambda^2\right\} \\
\mathcal{A} &= \kappa_1^2 \sigma_\lambda^2.
\end{aligned}$$

Hence

$$A_{1,m} = \frac{-\mathcal{B} \pm \sqrt{\mathcal{B}^2 - \mathcal{A}.\mathcal{C}}}{\mathcal{A}}.$$

We choose the negative value $A_{1,m}$ as in Dew-Becker et al. (2017). We provide a proof of Result IA.14 below. ■

Proof. Proof of Result IA.14. Let us find the physical moments and truncated moments:

$$\mathbb{E}_t\left(R_{M,t+1}^n\right) = \mathbb{E}_t\left(e^{nr_{m,t+1}}\right).$$

Note that

$$\begin{aligned} nr_{m,t+1} &= n\kappa_0 + n\left(\kappa_1 - 1\right)A_{0,m} + n\kappa_1 A_{1,m}\left(1-\rho_\lambda\right)\mu_\lambda \\ &\quad + n\left(\kappa_1\rho_\lambda - 1\right)A_{1,m}\lambda_t + n\kappa_1 A_{1,m}\sigma_\lambda\sqrt{\lambda_t}z_{\lambda,t+1} \\ &\quad + n\phi\mu_c + n\phi\sigma_c z_{c,t+1} + n\phi J_{t+1}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}_t\left(R_{M,t+1}^n\right) &= \exp\left\{\begin{array}{c} n\kappa_0 + n\left(\kappa_1 - 1\right)A_{0,m} + n\kappa_1 A_{1,m}\left(1-\rho_\lambda\right)\mu_\lambda \\ +n\phi\mu_c + \frac{1}{2}\left(n\phi\sigma_c\right)^2 \\ +\left\{n\left(\kappa_1\rho_\lambda - 1\right)A_{1,m} + \frac{1}{2}\left(n\kappa_1 A_{1,m}\sigma_\lambda\right)^2\right\}\lambda_t \end{array}\right\} \\ &\quad \times\mathbb{E}_t\left(\exp\left\{n\phi J_{t+1}\right\}\right). \end{aligned}$$

Note that

$$\mathbb{E}_t\left(\exp\left(\left(n\phi\right)J_{t+1}\right)\right) = \mathbb{E}_t\exp\left(\varphi_\phi N_{t+1}\right) = \exp\left(\lambda_t\left(e^{\varphi_\phi} - 1\right)\right),$$

where

$$\varphi_\phi = \left(n\phi\right)\mu_d + \frac{1}{2}\left(n\phi\right)^2\sigma_d^2.$$

Thus,

$$\mathbb{E}_t\left(R_{M,t+1}^n\right) = \exp\left\{\begin{array}{c} n\kappa_0 + n\left(\kappa_1 - 1\right)A_{0,m} + n\kappa_1 A_{1,m}\left(1-\rho_\lambda\right)\mu_\lambda \\ +n\phi\mu_c + \frac{1}{2}\left(n\phi\sigma_c\right)^2 \\ +\left\{n\left(\kappa_1\rho_\lambda - 1\right)A_{1,m} + \frac{1}{2}\left(n\kappa_1 A_{1,m}\sigma_\lambda\right)^2 + \left(e^{\varphi_\phi} - 1\right)\right\}\lambda_t \end{array}\right\}.$$

Now, let us find the truncated moments

$$\begin{aligned} \mathbb{E}_t\left(R_{M,t+1}^n 1_{R_{M,t+1}>a}\right) &= \mathbb{E}_t\left(e^{nr_{m,t+1}}1_{nr_{m,t+1}>n\log a}\right) \\ &= \mathbb{E}_t\left(e^{nr_{m,t+1}}1_{nr_{m,t+1}>n\log a}\right) \end{aligned}$$

Note that

$$nr_{m,t+1} = \mathcal{A}_t + Z_{t+1}$$

where

$$\begin{aligned}
\mathcal{A}_t &= n\kappa_0 + n(\kappa_1 - 1) A_{0,m} + n\kappa_1 A_{1,m}(1-\rho_\lambda)\mu_\lambda \\
&\quad + n(\kappa_1\rho_\lambda - 1) A_{1,m}\lambda_t + n\phi\mu_c \\
Z_{t+1} &= n\kappa_1 A_{1,m}\sigma_\lambda\sqrt{\lambda_t} z_{\lambda,t+1} + n\phi\sigma_c z_{c,t+1} + n\phi J_{t+1}
\end{aligned}$$

■

Proof. Let us compute the risk neutral moments

$$\begin{aligned}
\log \mathbb{E}_t^*\left(R_{M,t+1}^n\right) &= \log\left\{\mathbb{E}_t\left(\frac{M_{t+1}}{\mathbb{E}_t M_{t+1}} R_{M,t+1}^n\right)\right\} \\
&= \log R_{f,t+1} + \log\left\{\mathbb{E}_t\left(e^{m_{t+1}+nr_{m,t+1}}\right)\right\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
m_{t+1} + nr_{m,t+1} &= n\kappa_0 + n(\kappa_1 - 1) A_{0,m} + n\kappa_1 A_{1,m}(1-\rho_\lambda)\mu_\lambda \\
&\quad + n(\kappa_1\rho_\lambda - 1) A_{1,m}\lambda_t + n\kappa_1 A_{1,m}\sigma_\lambda\sqrt{\lambda_t} z_{\lambda,t+1} \\
&\quad + n\phi\mu_c + n\phi\sigma_c z_{c,t+1} + n\phi J_{t+1} \\
&\quad + A_0^{\mathrm{s}} + A_1^{\mathrm{s}}\lambda_t + A_2^{\mathrm{s}}\lambda_{t+1} + A_3^{\mathrm{s}} c_{t+1},
\end{aligned}$$

which simplifies to

$$\begin{aligned}
m_{t+1} + nr_{m,t+1} &= \left\{\begin{array}{c} n\kappa_0 + n(\kappa_1 - 1) A_{0,m} + n\kappa_1 A_{1,m}(1-\rho_\lambda)\mu_\lambda \\ \\ + A_0^{\mathrm{s}} + A_2^{\mathrm{s}}(1-\rho_\lambda)\mu_\lambda + n\phi\mu_c + A_3^{\mathrm{s}}\mu_c \end{array}\right\} \\
&\quad + \left\{n(\kappa_1\rho_\lambda - 1) A_{1,m} + A_1^{\mathrm{s}} + A_2^{\mathrm{s}}\rho_\lambda\right\}\lambda_t \\
&\quad + \left\{n\kappa_1 A_{1,m} + A_2^{\mathrm{s}}\right\}\sigma_\lambda\sqrt{\lambda_t} z_{\lambda,t+1} \\
&\quad + \left(n\phi + A_3^{\mathrm{s}}\right)\sigma_c z_{c,t+1} + \left(n\phi + A_3^{\mathrm{s}}\right) J_{t+1}.
\end{aligned}$$

Thus,

$$\mathbb{E}_t^*\left[R_{M,t\to t+1}^n\right] = R_{f,t+1}\exp\left(\begin{array}{c} \left\{\begin{array}{c} n\kappa_0 + n\left(\kappa_1 - 1\right)A_{0,m} + n\kappa_1 A_{1,m}\left(1-\rho_\lambda\right)\mu_\lambda \\ +A_0^{\mathrm{s}} + A_2^{\mathrm{s}}\left(1-\rho_\lambda\right)\mu_\lambda + n\phi\mu_c + A_3^{\mathrm{s}}\mu_c \end{array}\right\} \\ +\left\{n\left(\kappa_1\rho_\lambda - 1\right)A_{1,m} + A_1^{\mathrm{s}} + A_2^{\mathrm{s}}\rho_\lambda\right\}\lambda_t \\ +\frac{1}{2}\left(n\kappa_1 A_{1,m} + A_2^{\mathrm{s}}\right)^2\sigma_\lambda^2\lambda_t \\ +\frac{1}{2}\left(n\phi + A_3^{\mathrm{s}}\right)^2\sigma_c^2 + \lambda_t\left(e^{\varphi_{\phi,s}^*} - 1\right) \end{array}\right).$$

Since

$$\mathbb{E}_t\left(\exp\left(\left(n\phi + A_3^{\mathrm{s}}\right)J_{t+1}\right)\right) = \exp\left(\lambda_t\left(e^{\varphi_{\phi,s}^*} - 1\right)\right),$$

where

$$\varphi_{\phi,s}^* = \left(n\phi + A_3^{\mathrm{s}}\right)\mu_d + \frac{1}{2}\left(n\phi + A_3^{\mathrm{s}}\right)^2\sigma_d^2.$$

Now, let us compute the truncated risk neutral moment

$$\mathbb{E}_t^*\left(R_{M,t+1}^n 1_{R_{M,t+1}>a}\right) = R_{f,t+1}\mathbb{E}_t\left(e^{m_{t+1}+nr_{m,t+1}}1_{R_{M,t+1}>a}\right).$$

Note that

$$m_{t+1} + nr_{m,t+1} = Z_{t+1}^* + \mathcal{A}_t^*$$

where

$$Z_{t+1}^* = \left\{n\kappa_1 A_{1,m} + A_2^{\mathrm{s}}\right\}\sigma_\lambda\sqrt{\lambda_t}z_{\lambda,t+1} + \left(n\phi + A_3^{\mathrm{s}}\right)\sigma_c z_{c,t+1} + \left(n\phi + A_3^{\mathrm{s}}\right)J_{t+1}$$

and

$$\begin{aligned} \mathcal{A}_t^* &= \left\{\begin{array}{c} n\kappa_0 + n\left(\kappa_1 - 1\right)A_{0,m} + n.\kappa_1 A_{1,m}\left(1-\rho_\lambda\right)\mu_\lambda \\ +A_0^{\mathrm{s}} + A_2^{\mathrm{s}}\left(1-\rho_\lambda\right)\mu_\lambda + n\phi\mu_c + A_3^{\mathrm{s}}\mu_c \end{array}\right\} \\ &\quad +\left\{n\left(\kappa_1\rho_\lambda - 1\right)A_{1,m} + A_1^{\mathrm{s}} + A_2^{\mathrm{s}}\rho_\lambda\right\}\lambda_t. \end{aligned}$$

Given the state variable λ_t, the truncated risk-neutral moment can be computed at each date. ■

IA.8 Discussion of Differences with Respect to the Decomposition in Beason and Schreindorfer (2020)

Beason and Schreindorfer (2020) present a related decomposition of the *unconditional* equity risk premium that generates different implications for the relative contributions for downside, central, and upside risk premia to the total risk premium compared to our results. Namely, results in Beason and Schreindorfer (2020) imply that their equivalent of the downside, central, and upside risk premia constitute (approximately) 80%, 30%, and -10% of the total equity risk premium, respectively. This empirical result is very different than our baseline results (Table 2, Panel B), which imply the downside and central risk premia contribute similar amounts to the total risk premium (unconditionally) at about 45% each, and the upside risk premium contributes about 10% of the total risk premium unconditionally. To understand these different empirical results, we discuss three differences between our methodology and that in Beason and Schreindorfer (2020). First, Beason and Schreindorfer (2020) use a slightly different definition of the risk premium decomposition than we use herein. Second, their unconditional decomposition makes use of different empirical techniques to estimate the physical and risk-neutral market return densities than we use herein. Third, Jensen's inequality effects could explain some of the discrepancy between our respective decomposition results. We discuss each of these issues in more detail below.

We begin by discussing the relationship between our definition of the equity risk premium decomposition relative to that in Beason and Schreindorfer (2020). Starting with their Equation 4 and redefining their net market returns, R, in that equation to be gross returns (to be consistent with our consistent use of gross market returns), their truncated of risk premium associated with market returns between x_1 and x_2 is given by:

$$\widetilde{\mathbb{RP}}^{(1)}\left[A_{x_{12}}\right] \equiv \int_{x_1}^{x_2} R_M\left(f\left(R_M\right) - f^*\left(R_M\right)\right) dR_M. \tag{IA.86}$$

We use $\widetilde{\mathbb{RP}}^{(1)}\left[A_{x_{12}}\right]$ to designate this risk premium to highlight that this definition is analogous to (but not exactly the same as) our definition of the risk premium, $\mathbb{RP}^{(1)}\left[A_s\right]$, in Equation 29.[49] f and f^* represent the unconditional physical and risk-neutral return densities, respectively. To highlight the differences between the two risk premium definitions, we focus on the downside region for clarity. This corresponds to our case where $A_s = A_d$. The same case is obtained from Equation IA.86 by setting $x_1 = 0$ and $x_2 = \underline{x}$, which is the cutoff for our downside region (0.9). In this case, the Beason and Schreindorfer (2020) risk

[49] We remove the "$t \to T$" subscripts for simplicity since Beason and Schreindorfer (2020) consider only the one-month horizon.

premium definition becomes:

$$\widetilde{\mathbb{RP}}^{(1)}\left[A_d\right] \equiv \int_0^{\underline{x}} R_M\left(f\left(R_M\right)-f^*\left(R_M\right)\right) dR_M. \tag{IA.87}$$

Equivalently, we can express this integral in terms of expectation operators:

$$\widetilde{\mathbb{RP}}^{(1)}\left[A_d\right] \equiv \mathbb{E}\left[R_M \mathbb{I}_{A_d}\right]-\mathbb{E}^*\left[R_M \mathbb{I}_{A_d}\right]$$

where $\mathbb{I}_{A_d}$ is an indicator function for realizations of market returns in region A_d (that is, $R_M \in [0, \underline{x}]$). Using the identity $R_M \equiv R_M - R_f + R_f$, we can express the Beason and Schreindorfer (2020) downside risk premium in terms of a component that is equivalent to the unconditional version of our downside risk premium, $\mathbb{RP}^{(1)}\left[A_d\right]$, plus a component that is equivalent to a contingent claim that pays off one dollar in the event that $R_M \in A_d$:

$$\begin{aligned}
\widetilde{\mathbb{RP}}^{(1)}\left[A_d\right] &\equiv \mathbb{E}\left[\left(R_M-R_f+R_f\right) \mathbb{I}_{A_d}\right]-\mathbb{E}^*\left[\left(R_M-R_f+R_f\right) \mathbb{I}_{A_d}\right] \\
&= \mathbb{E}\left[\left(R_M-R_f\right) \mathbb{I}_{A_d}\right]-\mathbb{E}^*\left[\left(R_M-R_f\right) \mathbb{I}_{A_d}\right] \\
&\quad +\mathbb{E}\left[R_f \mathbb{I}_{A_d}\right]-\mathbb{E}^*\left[R_f \mathbb{I}_{A_d}\right] \\
&\equiv \mathbb{RP}^{(1)}\left[A_d\right]+R_f\left(\mathbb{E}\left[\mathbb{I}_{A_d}\right]-\mathbb{E}^*\left[\mathbb{I}_{A_d}\right]\right).
\end{aligned} \tag{IA.88}$$

Intuitively, this discrepancy exists because Beason and Schreindorfer (2020) define the truncated equity risk premium in terms of market returns in Equation IA.86, whereas we define our truncated risk premia in terms of excess market returns throughout. Note that when integrating Equation IA.86 over the entire return space ($x_1 = 0$, $x_2 \to \infty$), the two definitions are consistent since $\int_0^{\infty} R_M f^*\left(R_M\right) dR_M = R_f$. However, the two definitions are distinct when the region of interest is a subset of the return space. For completeness, the truncated market risk premia in the central and upside regions in Beason and Schreindorfer (2020) are related to ours as follows:

$$\widetilde{\mathbb{RP}}^{(1)}\left[A_c\right] \equiv \mathbb{RP}^{(1)}\left[A_c\right]+R_f\left(\mathbb{E}\left[\mathbb{I}_{A_c}\right]-\mathbb{E}^*\left[\mathbb{I}_{A_c}\right]\right) \text{ and} \tag{IA.89}$$

$$\widetilde{\mathbb{RP}}^{(1)}\left[A_u\right] \equiv \mathbb{RP}^{(1)}\left[A_u\right]+R_f\left(\mathbb{E}\left[\mathbb{I}_{A_u}\right]-\mathbb{E}^*\left[\mathbb{I}_{A_u}\right]\right) \tag{IA.90}$$

Therefore, the unconditional truncated market risk premia under our definition will be different than those implied by the Beason and Schreindorfer (2020) by terms related to the risk premia on contingent claims of the form $\mathbb{E}\left[\mathbb{I}_{A_s}\right]-\mathbb{E}^*\left[\mathbb{I}_{A_s}\right]$ that pay off one dollar in each respective region of market returns, $s \in \{d, c, u\}$. We call these "Arrow-Debreu risk premia" and define

$$\mathbb{RP}_t^{(0)}\left[A_s\right] \equiv \mathbb{E}_t\left[\mathbb{I}_{A_s}\right]-\mathbb{E}_t^*\left[\mathbb{I}_{A_s}\right].$$

How do we expect these differences highlighted in Equations IA.88-IA.90 to manifest in terms of the measured truncated risk premia? Using our methodology, we can compute the conditional Arrow-Debreu risk premia, $\mathbb{RP}_t^{(0)}[A_s]$, by setting $n = 0$ in Corollary 1 to estimate $\mathbb{E}[\mathbb{I}_A]$ and using standard techniques described in Internet Appendix IA.4 to estimate the risk-neutral counterpart. We plot estimated $\mathbb{RP}_t^{(0)}[A_s]$ in Figure IA.7. The unconditional (over time) average values of $\mathbb{RP}_t^{(0)}[A_s]$ for $s \in \{d, c, u\}$ are -1.88%, 1.35%, and 0.58% (annualized and in percent), respectively. Our unconditional average values of $\mathbb{RP}_t^{(1)}[A_s]$ for $s \in \{d, c, u\}$ (reported in Table 2 and Figure 2) are 4.45%, 3.33%, and 0.97%, respectively (annualized and in percent). Therefore, the discrepancy in the risk premium decomposition definitions could cause large differences in our measured $\mathbb{RP}^{(1)}[A_s]$ compared to $\widetilde{\mathbb{RP}}^{(1)}[A_s]$ implied by Beason and Schreindorfer (2020).

Our unconditional values of $\mathbb{RP}_t^{(0)}[A_s]$ imply the following relationships (all else equal) between our unconditional $\mathbb{RP}^{(1)}[A_s]$ and the $\widetilde{\mathbb{RP}}^{(1)}[A_s]$ values implied in Beason and Schreindorfer (2020):

$$\begin{aligned}
\widetilde{\mathbb{RP}}^{(1)}[A_d] &< \mathbb{RP}^{(1)}[A_d], \\
\widetilde{\mathbb{RP}}^{(1)}[A_c] &> \mathbb{RP}^{(1)}[A_c], \text{ and} \\
\widetilde{\mathbb{RP}}^{(1)}[A_u] &> \mathbb{RP}^{(1)}[A_u].
\end{aligned}$$

Figure 1 in Beason and Schreindorfer (2020) actually imply the opposite relationships between our respective truncated risk premia. Namely, their Figure 1 implies

$$\begin{aligned}
\widetilde{\mathbb{RP}}^{(1)}[A_d] &> \mathbb{RP}^{(1)}[A_d], \\
\widetilde{\mathbb{RP}}^{(1)}[A_c] &< \mathbb{RP}^{(1)}[A_c], \text{ and} \\
\widetilde{\mathbb{RP}}^{(1)}[A_u] &< \mathbb{RP}^{(1)}[A_u].
\end{aligned}$$

Why is this the case? This brings us to the second major difference between our decomposition and that in Beason and Schreindorfer (2020), which is related to how they estimate the physical and risk-neutral densities. In their case, they estimate f using realized historical returns and estimate f^* using an optimization approach over conditional f^* implied by option prices. Their this procedure yields an estimate for $\mathbb{E}[f^*]/\mathbb{E}[f]$ that is not monotonically decreasing in market returns (see Beason and Schreindorfer (2020), Figure 2).[50] The fact that their $\mathbb{E}[f^*]/\mathbb{E}[f]$ is only slightly decreasing in the central region yields an implied central risk premium contribution (approximately 30%) that is lower than that implied by our

[50]Note that the ratio $\mathbb{E}[f^*]/\mathbb{E}[f]$ is not exactly the same as the unconditional SDF, which is given by $\mathbb{E}[f^*/f]$.

methodology (approximately 45%). The fact that their $\mathbb{E}[f^*]/\mathbb{E}[f]$ is slightly increasing in the upside region yields an implied upside risk premium that is negative with a contribution to the total risk premium of approximately -10% (compared to our estimate of approximately 10%).

We sidestep the issue of estimating the physical and risk-neutral densities by using our transformation between risk-neutral and physical moments implied by Corollary 1. As can be see in Figure IA.3, our methodology implies conditional SDFs that are (approximately) monotonically decreasing in market returns. The result is that our downside, central, and upside risk premia are all positive so that each has a positive contribution to the overall risk premium. Finally, the implied magnitude of the downside risk premium in Beason and Schreindorfer (2020) is larger than our unconditional value because their $\mathbb{E}[f^*]/\mathbb{E}[f]$ (their Figure 2) is higher than our conditional SDFs in the downside region (see Figure IA.3). This implies that their $\mathbb{E}[f] - \mathbb{E}[f^*]$ is higher than our implied value in this region, yielding a larger downside risk premium.

The third potential contributor to differences between our risk premium contribution measures and those in Beason and Schreindorfer (2020) is related to Jensen's inequality. Beason and Schreindorfer (2020) compute their contributions effectively by integrating over $\mathbb{E}[f] - \mathbb{E}[f^*]$, whereas we first compute conditional contributions and average over these. One way to make our measures more comparable to theirs would be to compute contributions directly from the unconditional average risk premium levels. For instance, using results reported in Table 2, we could compute: $\mathbb{E}\left[\mathbb{RP}^{(1)}_{t\to T}[A_d]\right]/\mathbb{E}\left[\mathbb{RP}^{(1)}_{t\to T}[A]\right] = 4.45/8.72 \approx 51\%$. Since our estimate of the downside risk premium contribution reported in Table 2 is approximately 46%, this implies that even ignoring the Jensen's inequality terms would not reconcile the large differences between our risk premium contribution estimates and those in Beason and Schreindorfer (2020).

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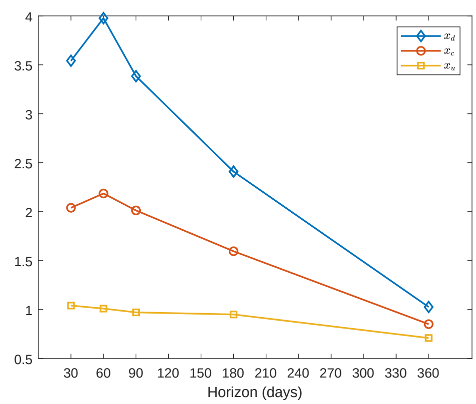


Figure IA.1

Relative Risk Aversion $(1/\tau\,(x_s))$

This graph plots estimates of relative risk aversion based on reported values of $\tau\,(x_s)$ in Table 1. Relative risk aversion is simply $1/\tau\,(x_s)$ (see Equation 14). Values are plotted for three points in the return space (x_d, x_c, and x_u) corresponding to the regions A_d, A_c, and A_u, respectively, and across five horizons (30, 60, 90, 180, and 360 days).

(a) $\tau(x_s)$ **(b) $\rho(x_s)$** **(c) $\kappa(x_s)$**

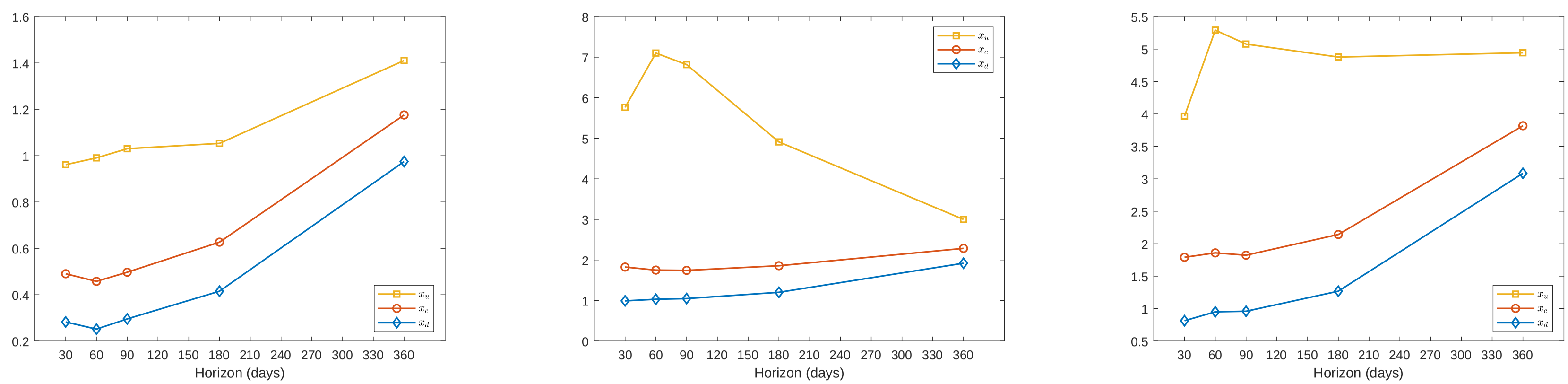


Figure IA.2
Estimated Preference Parameters

These graphs plot estimates of $\tau(x_s)$ (Panel (a)), $\rho(x_s)$ (Panel (b)), and $\kappa(x_s)$ (Panel (c)) based on reported values in Table 1. Values are plotted for three points in the return space (x_d, x_c, and x_u) corresponding to the regions A_d, A_c, and A_u, respectively, and across five horizons (30, 60, 90, 180, and 360 days).

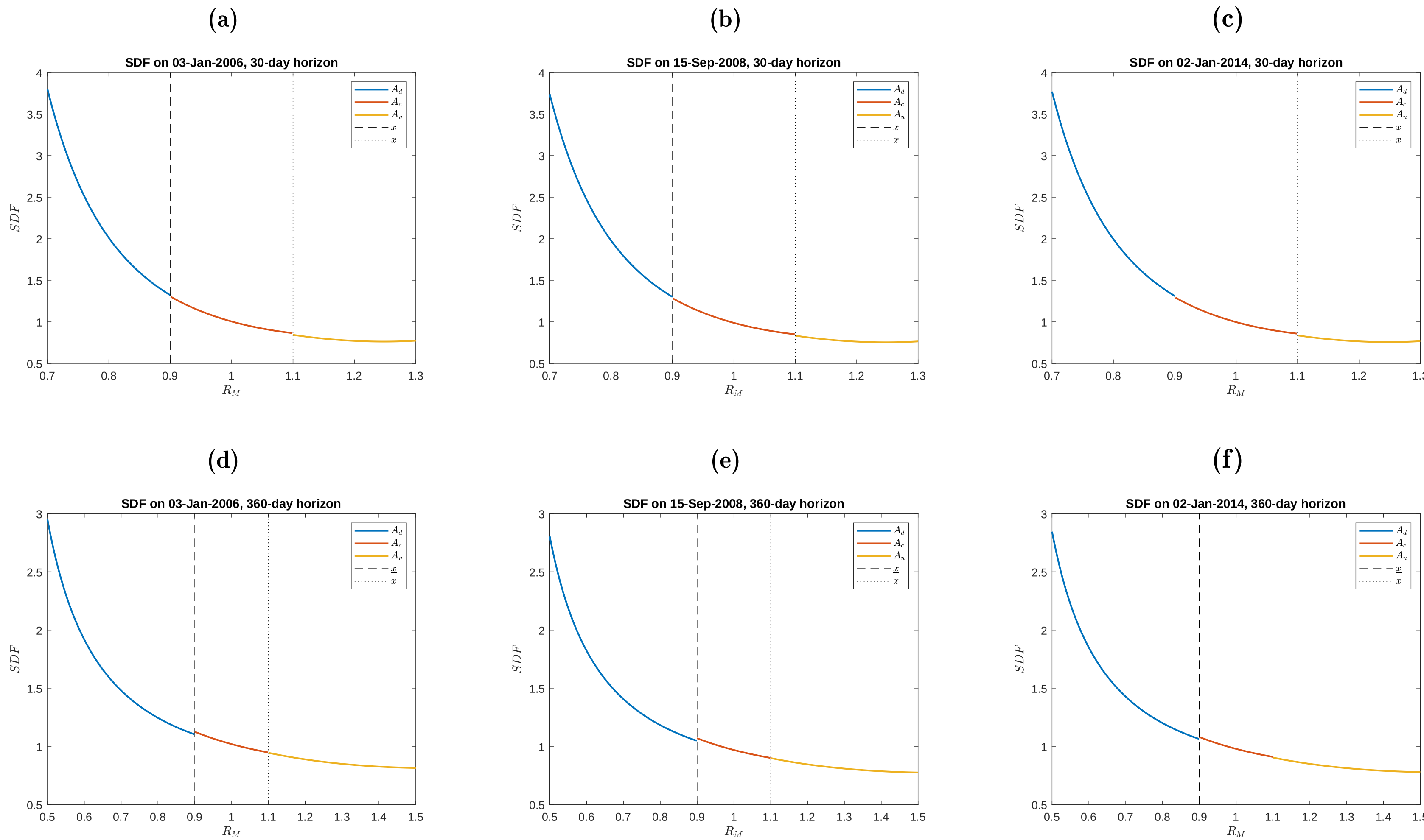
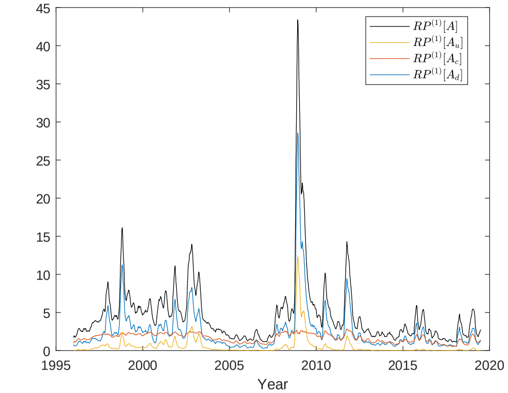


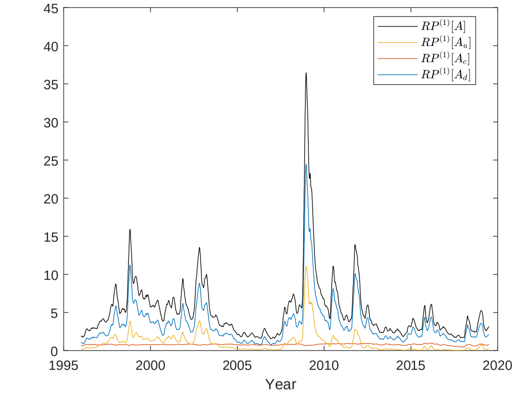
Figure IA.3
Implied SDF at Selected Dates

These graphs plot estimated values of the SDF across three regions of interest (A_d, A_c, and A_u) with region boundaries at $\underline{x} = 0.9$ and $\overline{x} = 1.1$ according to the inverse of the terms in Equation 18 using estimated preference parameters from Table 1. The SDF is estimated at two horizons (30 days - Panels (a), (b), and (c); and 360 days - Panels (d), (e), and (f)), and on three dates (3 January, 2006 - Panels (a) and (d); 15 September, 2008 - Panels (b) and (e); and 2 January, 2014 - Panels (c) and (f)). The plots also make use of $\mathbb{M}^{*(n)}_{t\to T}[A_s]$ estimates from options prices on these dates.

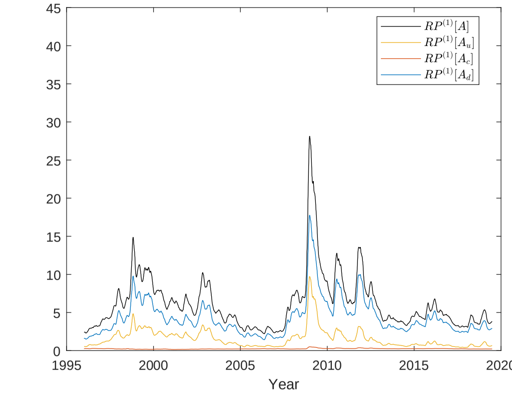
(a) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Levels (30-Day Horizon)



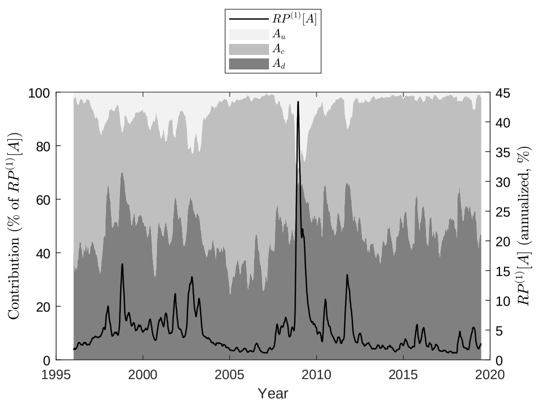
(b) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Levels (90-Day Horizon)



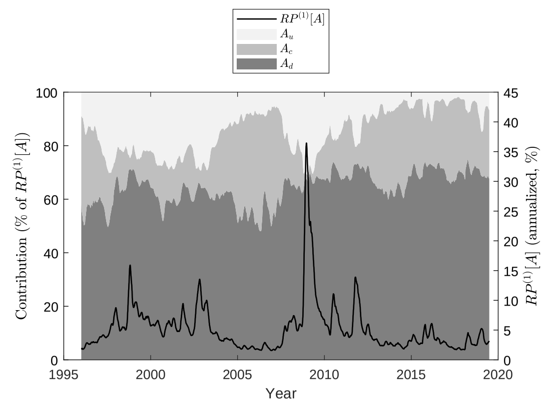
(c) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Levels (360-Day Horizon)



(d) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Contribs. (30-Day Horizon)



(e) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Contribs. (90-Day Horizon)



(f) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Contribs. (360-Day Horizon)

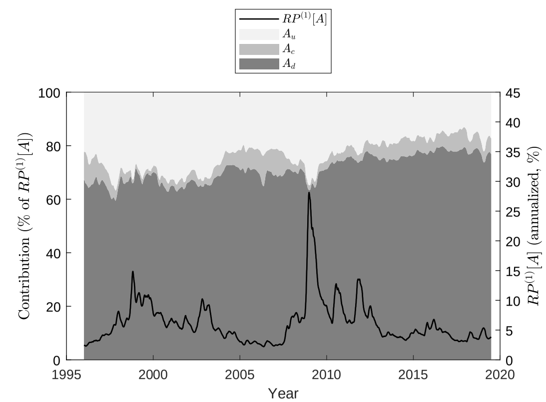


Figure IA.4

Data-Implied Market Risk Premium Decomposition (Restricted Preference Parameters)

These graphs plot the data-implied restricted risk premium decompositions from Section 2.3 based on Proposition 3 with $n = 1$ (i.e., the market risk premium). The decompositions set preference parameters to be $\tau = 1$, $\rho = 2$, and $\kappa = 4$ across all regions and horizons. Panels (a)-(c) plot the annualized risk premium levels at each date in percent. Panels (d)-(e) plot each component's contribution to the total risk premium at each date as a fraction of the total risk premium. The dark/medium/light shaded regions represent the downside/central/upside risk premium contributions, respectively. The decompositions are computed at three horizons (30 days - Panels (a) and (d); 90 days - Panels (b) and (e); and 360 days - Panels (c) and (f)) and use A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$. All time series are smoothed by averaging over two months of lagged daily data to reduce the appearance of noise.

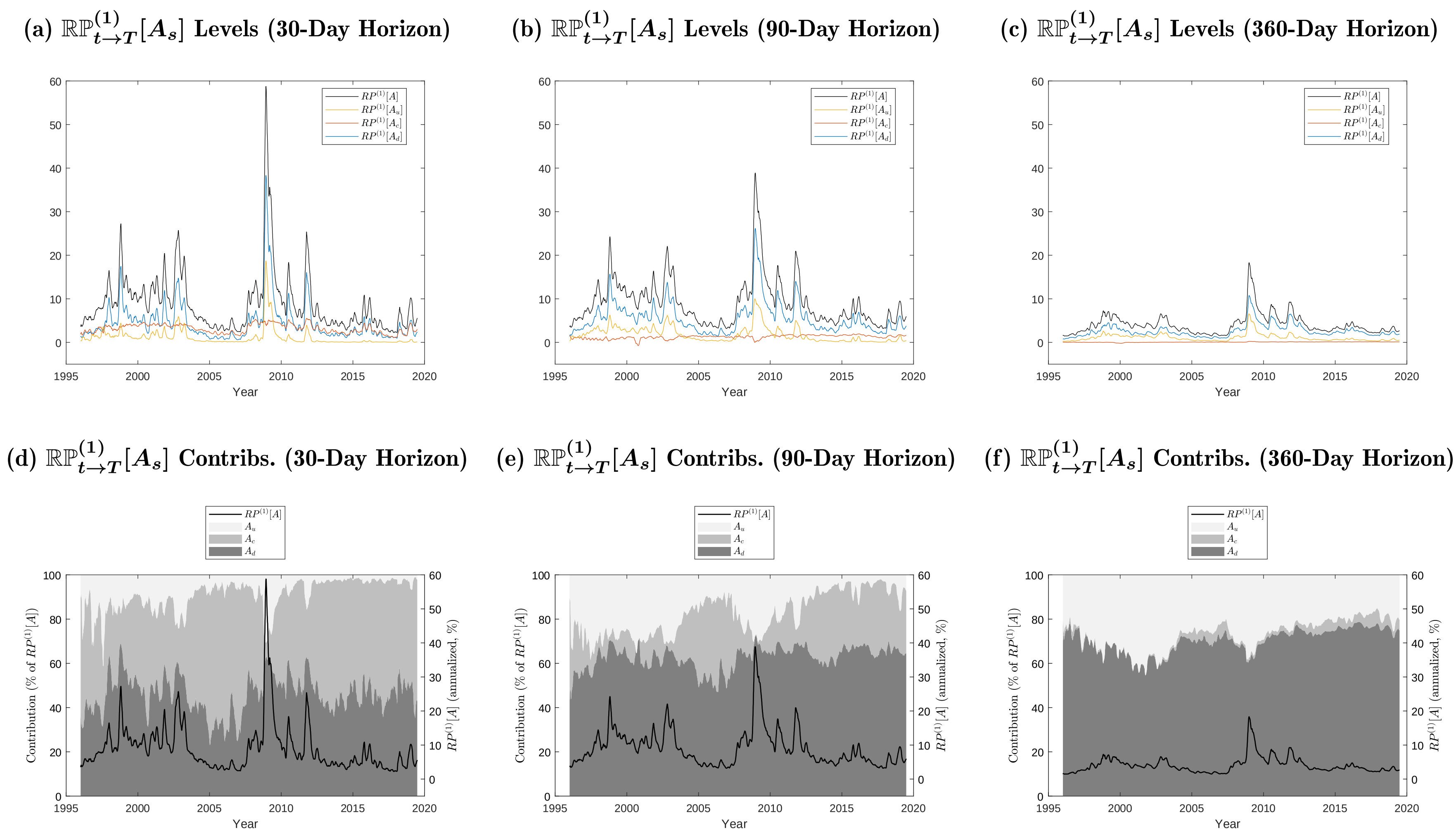
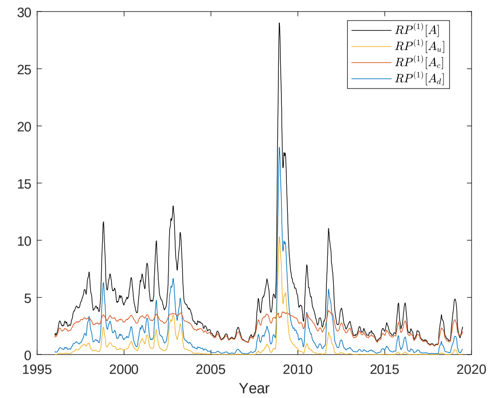


Figure IA.5

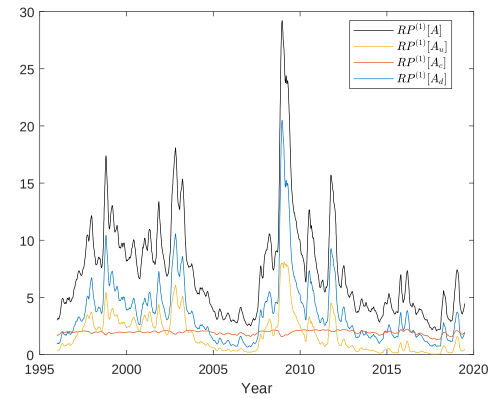
Data-Implied Market Risk Premium Decomposition (Unrestricted Preference Parameters; Observed Prices)

These graphs plot the data-implied unrestricted risk premium decompositions from Section 2.2 based on Proposition 3 with $n = 1$ (i.e., the market risk premium). The decompositions use risk-neutral moments computed by numerically integration over observed option prices directly rather than using the implied volatility fitting method. Preference parameters are also re-estimated using these moments. Panels (a)-(c) plot the annualized risk premium levels at each date. Panels (d)-(e) plot each component's contribution to the total risk premium at each date as a fraction of the total risk premium. The dark/medium/light shaded regions represent the downside/central/upside risk premium contributions, respectively, and are measured on the left vertical axes. The decompositions are computed at three horizons (30 days - Panels (a) and (d); 90 days - Panels (b) and (e); and 360 days - Panels (c) and (f)) and use A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$. All time series are smoothed by averaging over two months of lagged daily data to reduce the appearance of noise.

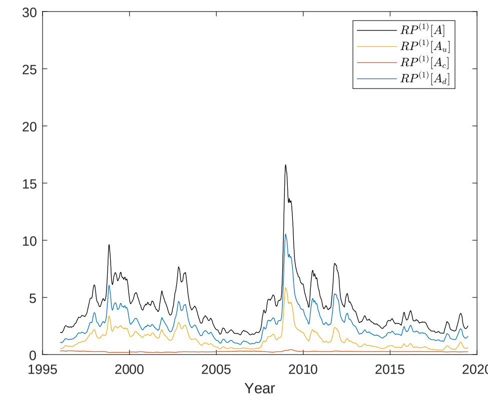
(a) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Levels (30-Day Horizon)



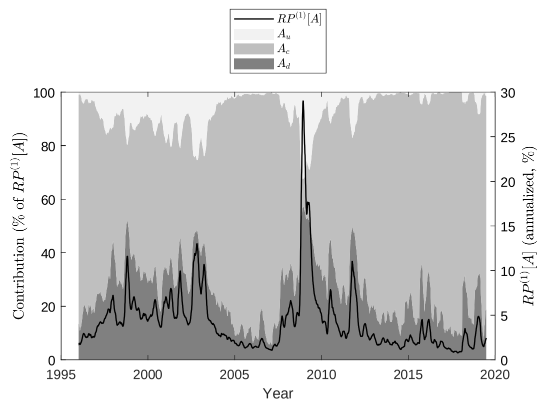
(b) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Levels (90-Day Horizon)



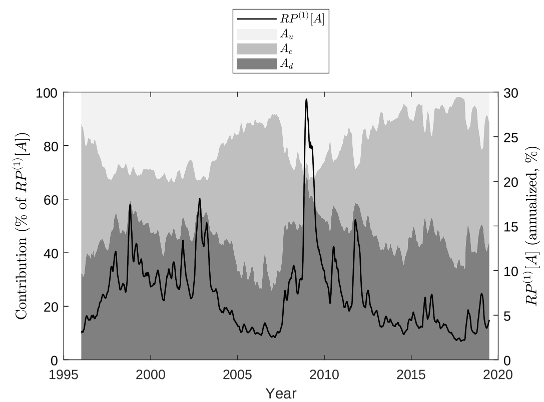
(c) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Levels (360-Day Horizon)



(d) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Contribs. (30-Day Horizon)



(e) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Contribs. (90-Day Horizon)



(f) $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ Contribs. (360-Day Horizon)

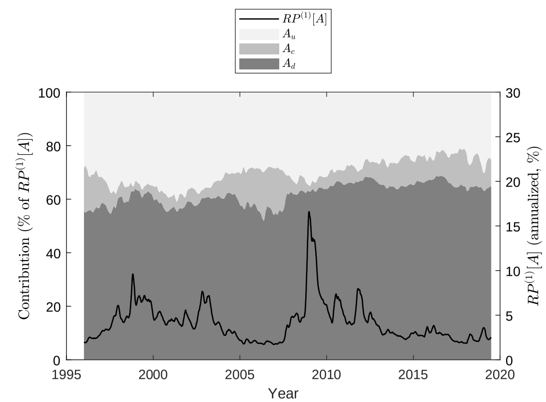


Figure IA.6

Data-Implied Market Risk Premium Decomposition (Unrestricted Preference Parameters; Overpriced Option Adjustment)

These graphs plot the data-implied unrestricted risk premium decompositions from Section 2.2 based on Proposition 3 with $n = 1$ (i.e., the market risk premium). The decompositions use risk-neutral moments computed using the mispricing adjustment described in Subsection IA.2.4. Preference parameters are also re-estimated using these moments. Panels (a)-(c) plot the annualized risk premium levels at each date. Panels (d)-(e) plot each component's contribution to the total risk premium at each date as a fraction of the total risk premium. The dark/medium/light shaded regions represent the downside/central/upside risk premium contributions, respectively, and are measured on the left vertical axes. The decompositions are computed at three horizons (30 days - Panels (a) and (d); 90 days - Panels (b) and (e); and 360 days - Panels (c) and (f)) and use A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$. All time series are smoothed by averaging over two months of lagged daily data to reduce the appearance of noise.

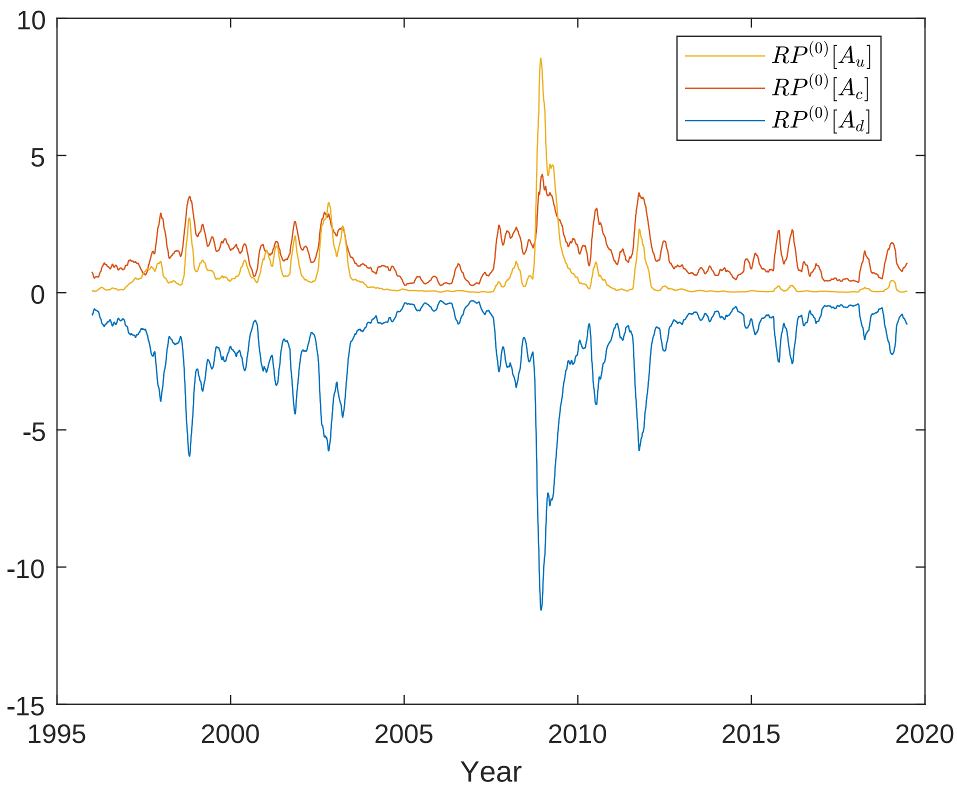


Figure IA.7
Arrow-Debreu Risk Premia

This graph plots the data-implied Arrow-Debreu risk premia of the form $\mathbb{RP}_t^{(0)}[A_s] \equiv \mathbb{E}_t[\mathbb{I}_{A_s}] - \mathbb{E}_t^*[\mathbb{I}_{A_s}]$ estimated using unrestricted preference parameters reported in Table 1 to estimate $\mathbb{E}_t[\mathbb{I}_{A_s}]$ according to Corollary 1 with $n = 0$ and $\mathbb{E}_t^*[\mathbb{I}_{A_s}]$ according to standard techniques described in Internet Appendix IA.4. These are computed at the 30-day horizon and annualized to be consistent with plots of our market risk premium in Figure 2. The decompositions use A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$. All time series are smoothed by averaging over two months of lagged daily data to reduce the appearance of noise.

Table IA.1

Forecasting Regressions Using Restricted Data-Implied Risk Premia

This table reports excess market return forecasting regression results based on the specification in Equation 33 using risk premia from the restricted decomposition described in Subsection 2.3. Results under "Full Sample" use data from January, 1996 through June, 2019. Results under "Ex Crisis" use the same time period except they exclude data from August, 2008 through January, 2009 (i.e., data from the height of the 2008 Financial Crisis). Results are reported for 30, 90, and 360-day forecast horizons. Excess market returns are measured as ex dividend returns on the S&P 500 index obtained from CRSP less the risk-free rate obtained from Kenneth French's website. Data is daily and excess returns at each horizon are computed by compounding daily returns to the horizon of interest and subtracting the compounded risk-free rate. T-statistics are reported in parentheses and are computed according to Newey and West (1987) with lag values according to Newey and West (1994), with one slight modification. Since we have overlapping data, we multiply the Newey and West (1994)-implied lag value by the number of trading days in each horizon and use this as our lag value. R^2_{IS} indicates standard adjusted in-sample R-squared values. $R^2_{pseudoOS}$ corresponds to out-of-sample R-squared values computed according to the methodology in Goyal and Welch (2008) with two slight modifications. First, they are labeled as "pseudo" out-of-sample because they use preference parameters estimated using our full sample of data. Second, they require no estimation and just use our ex ante measure for the total risk premium as the model-implied market return forecast. Historical average returns are estimated using ex dividend S&P 500 excess returns starting in 1926 and obtained from CRSP. We do not report intercepts or $R^2_{pseudoOS}$ values for the truncated risk premia since they do not have the theoretical implications that $a_T = 0$ and $b_T = 1$ (as is the case for the total risk premium).

$$R_{M,t\to T} - R_{f,t\to T} = a_T + b_T \mathbb{RP}^{(1)}[A_s] + \varepsilon_{t\to T}$$

	Full Sample												Ex Crisis											
Horizon (days)	30				90				360				30				90				360			
	I	*II*	*III*	*IV*	*I*	*II*	*III*	*IV*	*I*	*II*	*III*	*IV*	*I*	*II*	*III*	*IV*	*I*	*II*	*III*	*IV*	*I*	*II*	*III*	*IV*
a_T	4.17	-	-	-	2.94	-	-	-	0.63	-	-	-	-4.34	-	-	-	-4.79	-	-	-	1.75	-	-	-
	(1.02)	-	-	-	(0.71)	-	-	-	(0.13)	-	-	-	(-1.16)	-	-	-	(-1.16)	-	-	-	(0.36)	-	-	-
$\mathbb{RP}^{(1)}_{t\to T}$	0.39				0.51				0.85				2.86				2.58				1.01			
	(0.40)				(0.59)				(2.03)				(4.09)				(4.13)				(1.67)			
$\mathbb{RP}^{(1)}_{t\to T}[A_d]$		0.33				0.62				0.97				2.29				2.55				1.14		
		(0.43)				(0.73)				(2.11)				(4.19)				(4.60)				(1.87)		
$\mathbb{RP}^{(1)}_{t\to T}[A_c]$			0.13				-1.71				5.99				2.35				-2.19				6.59	
			(0.07)				(-0.64)				(4.73)				(1.52)				(-0.75)				(3.74)	
$\mathbb{RP}^{(1)}_{t\to T}[A_u]$				0.15				0.13				0.47				1.40				1.29				0.50
				(0.38)				(0.26)				(1.26)				(5.07)				(2.91)				(0.85)
R^2_{IS}	0.00	0.00	0.00	0.00	0.01	0.01	0.00	0.00	0.04	0.04	0.25	0.02	0.03	0.03	0.00	0.04	0.09	0.10	0.00	0.07	0.05	0.06	0.30	0.02
R^2_{OS}	0.00	-	-	-	-0.01	-	-	-	-0.03	-	-	-	0.02	-	-	-	0.07	-	-	-	0.07	-	-	-

Table IA.2
Unrestricted Data-Implied Market Risk Premium Decomposition Summary Statistics (Observed Prices)

This table reports summary statistics for the unrestricted data-implied risk premium decomposition according to Proposition 3 with $n = 1$ (i.e., the market risk premium). The decompositions use risk-neutral moments computed by numerically integration over observed option prices directly rather than using the implied volatility fitting method. Preference parameters are also re-estimated using these moments. Panel A reports statistics for the risk premium levels (annualized, in percent) and Panel B reports statistics for the contributions of risk premia from each region to the total risk premium (as fractions of the total risk premium, in percent). A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$ and these labels correspond to the downside, central, and upside risk premia, respectively. $A = A_d \cup A_c \cup A_u$ and this label corresponds to the total risk premium. Statistics reported under "Unconditional" use the full estimated time series for each risk premium measure. Statistics reported under "Cond. Means" report the means for each time series conditional on 30-day risk-neutral variance ($\mathbb{M}^{*(2)}_{t\to T}[A]$) falling below it's first quartile ("Lo"), between its first and third quartiles ("Mid"), or above its third quartile ("Hi"). These correspond to periods of low, moderate, or high market volatility, respectively. Statistics are reported for risk premium decompositions at 30, 60, 90, 180, and 360-day horizons, and are based on daily data from January, 1996 through June, 2019.

		Panel A: $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ (%)					Panel B: $\mathbb{RP}^{(1)}_{t\to T}[A_s]/\mathbb{RP}^{(1)}_{t\to T}[A]$ (%)				
		Cond. Means			Unconditional		Cond. Means			Unconditional	
Horizon (days)	**Region**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**
30	$\boldsymbol{A}$	3.28	7.10	18.22	8.93	7.59					
	$\boldsymbol{A_d}$	1.22	3.17	10.36	4.48	4.93	37.53	44.53	55.80	45.60	9.76
	$\boldsymbol{A_c}$	1.88	3.21	4.42	3.18	1.14	58.26	47.47	29.09	45.57	13.89
	$\boldsymbol{A_u}$	0.13	0.60	3.22	1.14	2.22	4.21	8.00	15.11	8.83	8.22
60	$\boldsymbol{A}$	4.07	8.18	18.57	9.75	6.96					
	$\boldsymbol{A_d}$	2.03	4.52	11.43	5.63	4.69	50.96	57.04	63.24	57.07	7.59
	$\boldsymbol{A_c}$	1.67	2.20	2.05	2.03	0.59	43.13	29.59	13.59	28.97	13.00
	$\boldsymbol{A_u}$	0.23	1.12	4.34	1.70	2.19	5.91	13.37	23.17	13.96	10.09
90	$\boldsymbol{A}$	4.25	7.94	16.45	9.14	5.73					
	$\boldsymbol{A_d}$	2.41	4.75	10.30	5.55	3.81	58.64	62.78	66.00	62.55	6.02
	$\boldsymbol{A_c}$	1.34	1.38	0.92	1.25	0.49	33.43	19.57	7.10	19.92	11.74
	$\boldsymbol{A_u}$	0.31	1.39	4.20	1.83	1.84	7.93	17.65	26.90	17.53	10.29
180	$\boldsymbol{A}$	4.10	6.99	13.15	7.81	4.41					
	$\boldsymbol{A_d}$	2.67	4.52	8.58	5.07	3.04	68.96	69.54	69.44	69.37	4.27
	$\boldsymbol{A_c}$	0.62	0.41	0.19	0.40	0.26	16.46	7.07	1.56	8.04	6.72
	$\boldsymbol{A_u}$	0.55	1.55	3.46	1.78	1.30	14.58	23.39	29.00	22.59	7.68
360	$\boldsymbol{A}$	2.34	3.69	6.63	4.09	2.44					
	$\boldsymbol{A_d}$	1.60	2.41	4.04	2.62	1.52	73.77	71.37	65.35	70.47	5.76
	$\boldsymbol{A_c}$	0.11	0.06	0.06	0.07	0.07	5.13	1.97	0.72	2.44	2.43
	$\boldsymbol{A_u}$	0.45	0.91	2.07	1.08	0.88	21.10	26.66	33.93	27.09	7.22

Table IA.3
Unrestricted Data-Implied Market Risk Premium Decomposition Summary Statistics (Mispricing Adjustment)

This table reports summary statistics for the unrestricted data-implied risk premium decomposition according to Proposition 3 with $n = 1$ (i.e., the market risk premium). The decompositions use risk-neutral moments computed using the mispricing adjustment described in Subsection IA.2.4. Preference parameters are also re-estimated using these moments. Panel A reports statistics for the risk premium levels (annualized, in percent) and Panel B reports statistics for the contributions of risk premia from each region to the total risk premium (as fractions of the total risk premium, in percent). A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$ and these labels correspond to the downside, central, and upside risk premia, respectively. $A = A_d \cup A_c \cup A_u$ and this label corresponds to the total risk premium. Statistics reported under "Unconditional" use the full estimated time series for each risk premium measure. Statistics reported under "Cond. Means" report the means for each time series conditional on 30-day risk-neutral variance ($\mathbb{M}^{*(2)}_{t\to T}[A]$) falling below it's first quartile ("Lo"), between its first and third quartiles ("Mid"), or above its third quartile ("Hi"). These correspond to periods of low, moderate, or high market volatility, respectively. Statistics are reported for risk premium decompositions at 30, 60, 90, 180, and 360-day horizons, and are based on daily data from January, 1996 through June, 2019.

		Panel A: $\mathbb{RP}^{(1)}_{t\to T}[A_s]$ (%)					Panel B: $\mathbb{RP}^{(1)}_{t\to T}[A_s]/\mathbb{RP}^{(1)}_{t\to T}[A]$ (%)				
		Cond. Means			Unconditional		Cond. Means			Unconditional	
Horizon (days)	**Region**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**
30	A	1.43	3.37	8.80	4.24	3.77					
	A_d	0.17	0.87	4.05	1.49	2.28	10.79	23.13	40.19	24.31	12.31
	A_c	1.32	2.44	3.39	2.40	0.84	88.82	72.08	43.38	69.09	19.00
	A_u	0.01	0.20	1.82	0.56	1.25	0.39	4.79	16.43	6.60	7.35
60	A	2.80	6.03	13.77	7.16	5.20					
	A_d	0.71	2.35	7.28	3.17	3.38	25.91	38.28	49.58	38.01	10.11
	A_c	1.87	2.64	2.85	2.50	0.45	70.90	47.73	23.52	47.47	19.00
	A_u	0.09	0.94	3.94	1.48	1.91	3.19	13.99	26.91	14.52	10.10
90	A	3.19	6.38	13.20	7.29	4.59					
	A_d	1.16	2.97	7.39	3.63	3.03	36.34	45.87	53.82	45.48	8.09
	A_c	1.72	2.04	1.94	1.93	0.19	56.13	34.78	16.36	35.51	16.07
	A_u	0.24	1.34	4.02	1.73	1.71	7.53	19.35	29.83	19.01	9.72
180	A	3.27	5.92	11.38	6.62	3.88					
	A_d	1.73	3.37	6.94	3.85	2.60	52.86	56.70	59.83	56.52	5.22
	A_c	0.92	0.88	0.77	0.87	0.08	29.33	16.49	7.57	17.47	9.05
	A_u	0.58	1.64	3.68	1.88	1.40	17.81	26.82	32.59	26.01	7.01
360	A	2.25	3.75	6.84	4.15	2.33					
	A_d	1.39	2.32	4.25	2.57	1.51	61.94	61.98	61.49	61.85	4.10
	A_c	0.27	0.26	0.25	0.26	0.04	12.67	7.71	3.99	8.02	3.97
	A_u	0.56	1.15	2.35	1.30	0.87	25.39	30.31	34.51	30.13	4.85

Table IA.4
Utility-Implied Market Risk Premium Decomposition Average Values

This table reports average values for components of risk premium decomposition according to Proposition 3 assuming the representative investor has a specific utility function, and uses results from Section IA.3. Panel A reports average values for the risk premium levels (annualized) and Panel B reports average contributions of risk premia from each region to the total risk premium (as fractions of the total risk premium). A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$ and these labels correspond to the downside, central, and upside risk premia, respectively. $A = A_d \cup A_c \cup A_u$ and this label corresponds to the total risk premium. Statistics are reported for risk premium decompositions at 30, 60, 90, 180, and 360-day horizons, and are based on daily data from January, 1996 through June, 2019.

		Preference Assumption									
		Log	CRRA			CARA			HARA		
Horizon (days)	Region		$\alpha = 3$	$\alpha = 5$	$\alpha = 7$	$\alpha = 3$	$\alpha = 5$	$\alpha = 7$	RRA= 1	RRA= 1.1	RRA= 1.2
Panel A: Average Market Risk Premium Levels ($\mathbb{RP}^{(1)}_{t\to T}[A_s]$)											
30	A	4.13	11.19	17.30	22.86	11.03	17.38	23.38	4.13	4.54	4.95
	A_d	1.93	4.59	6.27	7.37	4.31	6.00	7.15	1.95	2.15	2.34
	A_c	1.63	4.73	7.65	10.35	4.71	7.63	10.31	1.60	1.77	1.93
	A_u	0.56	1.85	3.34	5.10	2.00	3.72	5.87	0.57	0.63	0.69
60	A	4.24	11.03	16.63	21.58	10.89	16.91	22.64	4.20	4.62	5.04
	A_d	2.32	5.34	7.16	8.29	5.01	6.89	8.09	2.31	2.54	2.77
	A_c	1.05	2.97	4.68	6.14	2.93	4.62	6.05	1.02	1.13	1.23
	A_u	0.84	2.66	4.71	7.05	2.89	5.32	8.38	0.84	0.93	1.01
90	A	4.28	10.85	16.14	20.79	10.78	16.68	22.52	4.23	4.65	5.07
	A_d	2.50	5.58	7.35	8.40	5.24	7.09	8.23	2.47	2.72	2.97
	A_c	0.74	2.01	3.09	3.93	1.97	3.02	3.81	0.70	0.77	0.84
	A_u	1.01	3.19	5.60	8.34	3.51	6.47	10.34	1.03	1.13	1.23
180	A	4.36	10.37	15.00	19.10	10.51	16.19	22.22	4.25	4.67	5.09
	A_d	2.70	5.62	7.11	7.90	5.30	6.93	7.81	2.64	2.90	3.17
	A_c	0.36	0.85	1.17	1.31	0.80	1.07	1.16	0.32	0.35	0.39
	A_u	1.27	3.83	6.62	9.76	4.33	8.08	13.12	1.26	1.38	1.51
360	A	4.47	9.83	13.96	17.90	10.48	16.66	24.29	4.24	4.66	5.08
	A_d	2.82	5.32	6.42	6.92	5.08	6.35	6.92	2.68	2.94	3.21
	A_c	0.16	0.25	0.19	0.03	0.18	0.06	-0.16	0.14	0.15	0.17
	A_u	1.46	4.19	7.25	10.85	5.15	10.16	17.44	1.39	1.53	1.67

Table IA.4
Utility-Implied Market Risk Premium Decomposition Average Values (continued)

		Preference Assumption									
		Log	CRRA			CARA			HARA		
Horizon (days)	Region		$\alpha = 3$	$\alpha = 5$	$\alpha = 7$	$\alpha = 3$	$\alpha = 5$	$\alpha = 7$	$RRA = 1$	$RRA = 1.1$	$RRA = 1.2$
Panel B: Average Market Risk Premium Contributions ($\mathbb{RP}^{(1)}_{t\to T}[A_s]/\mathbb{RP}^{(1)}_{t\to T}[A]$, in Percent)											
	A_d	41.83	37.05	32.99	29.61	35.38	31.66	28.47	42.41	42.34	42.29
30	A_c	50.44	53.53	55.92	57.48	54.35	56.24	57.39	49.67	49.75	49.81
	A_u	7.74	9.43	11.10	12.92	10.27	12.10	14.14	7.92	7.91	7.90
	A_d	52.64	46.98	42.18	38.07	44.92	40.41	36.39	52.90	52.87	52.85
60	A_c	33.83	36.42	38.27	39.26	37.00	38.29	38.81	33.22	33.28	33.32
	A_u	13.53	16.60	19.55	22.67	18.09	21.29	24.80	13.88	13.85	13.83
	A_d	57.73	51.50	46.20	41.61	49.16	44.03	39.35	58.01	57.98	57.95
90	A_c	24.18	25.86	27.11	27.49	26.12	26.82	26.70	23.03	23.14	23.23
	A_u	18.09	22.64	26.69	30.90	24.72	29.15	33.96	18.95	18.88	18.82
	A_d	63.05	55.88	49.72	44.25	52.90	46.48	40.35	63.23	63.20	63.18
180	A_c	11.41	11.60	11.51	10.75	11.28	10.74	9.58	10.33	10.43	10.51
	A_u	25.54	32.51	38.77	44.99	35.81	42.79	50.07	26.44	26.37	26.31
	A_d	64.80	56.67	49.37	42.62	52.29	43.49	34.81	64.82	64.81	64.80
360	A_c	4.75	3.74	2.47	1.06	2.93	1.43	0.01	4.27	4.32	4.35
	A_u	30.45	39.59	48.16	56.32	44.77	55.07	65.18	30.91	30.87	30.84

Table IA.5
Data- versus Model-Implied State Variable Processes

This table reports summary statistics for state variable processes extracted from the data (according to the methodology in Section 3) and based on model simulations. Panels A and B reports results for the Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012) models, respectively. Panel C reports results for the Bollerslev, Tauchen, and Zhou (2009) model. Panel D reports results for the Drechsler and Yaron (2011) model. Panels E and F report results for the Bekaert, Engstrom, and Ermolov (2020) models with and without preference shocks, respectively. Panel G reports results for the Gabaix (2012) model. Panel H reports results for the Wachter (2013) model. Models are simulated at the monthly frequency for 100 million periods and used to compute the "Simulation-implied" statistics. State variables extracted from the data are available at the daily frequency. To be consistent with model simulations, we compute 21 sets of each data-implied statistic from non-overlapping daily data sampled every 21 days (since there approximately 21 trading days in each calendar month). The "Data-implied" statistics are averages across each statistic from these 21 sets. 95% confidence intervals on the simulated statistics are computed to correspond to confidence intervals we expect to see under the model null given a random sample of 282 months (which corresponds to the number of months we observe in our data from January, 1996 through June, 2019). They are based on randomly sampling 10,000 sets of 282 months of data from the full 100 million simulated months for each model. The confidence intervals answer the question: "With 95% confidence, would we expect to observe our data-implied statistics under the model null?" "SV1" and "SV2" under the Correlation heading correspond with the first and second state variables from each model ordered according to their appearance in the first column.

Variable	Source	Mean	St. Dev.	Autocorr.	Corr. SV1	Corr. SV2
Panel A: Bansal and Yaron (2004)						
x_t	**Data-implied**	-2.1E-6	2.6E-3	0.91		
	Simulation-implied	9.7E-6	1.4E-3	0.96		
	[95% CI]	[-1.7E-3, 1.8E-3]	[8.2E-4, 2.2E-3]	[0.92, 0.99]		
σ_t^2	**Data-implied**	6.1E-5	8.4E-5	0.80	0.56	
	Simulation-implied	6.1E-5	1.1E-5	0.97	0.00	
	[95% CI]	[4.3E-5, 7.8E-5]	[6.0E-6, 1.8E-5]	[0.93, 0.99]	[-0.64, 0.65]	
Panel B: Bansal, Kiku, and Yaron (2012)						
x_t	**Data-implied**	-3.3E-5	5.5E-3	0.85		
	Simulation-implied	8.3E-6	1.2E-3	0.96		
	[95% CI]	[-1.4E-3, 1.5E-3]	[4.3E-4, 2.3E-3]	[0.91, 0.99]		
σ_t^2	**Data-implied**	7.3E-5	6.5E-5	0.80	0.81	
	Simulation-implied	7.3E-5	1.6E-5	0.98	0.00	
	[95% CI]	[1.2E-5, 1.7E-4]	[7.5E-6, 3.2E-5]	[0.93, 1.00]	[-0.67, 0.68]	
Panel C: Bollerslev, Tauchen, and Zhou (2009)						
$\sigma_{g,t}^2$	**Data-implied**	-4.6E-5	8.3E-3	0.78		
	Simulation-implied	4.6E-6	4.4E-3	0.96		
	[95% CI]	[-5.4E-3, 5.4E-3]	[2.5E-3, 7.6E-3]	[0.91, 0.99]		
q_t	**Data-implied**	1.4E-6	2.8E-5	0.97	-0.73	
	Simulation-implied	1.3E-6	1.7E-6	0.76	0.00	
	[95% CI]	[7.9E-7, 2.0E-6]	[9.8E-7, 2.8E-6]	[0.60, 0.88]	[-0.34, 0.34]	

Table IA.5

Data- versus Model-Implied State Variable Processes (continued)

Variable	Source	Mean	St. Dev.	Autocorr.	Corr. SV1	Corr. SV2
Panel D: Drechsler and Yaron (2011)						
x_t	**Data-implied**	5.6E-6	2.7E-3	0.80		
	Simulation-implied	-2.3E-6	1.1E-3	0.96		
	[95% CI]	[-1.4E-3, 1.3E-3]	[4.8E-4, 2.2E-3]	[0.90, 0.99]		
$\overline{\sigma}_t^2$	**Data-implied**	1.20	1.50	0.90	0.79	
	Simulation-implied	1.05	0.42	0.97	0.00	
	[95% CI]	[0.51, 1.69]	[0.26, 0.68]	[0.92, 0.99]	[-0.61, 0.63]	
σ_t^2	**Data-implied**	1.11	1.80	0.79	1.00	0.79
	Simulation-implied	1.07	1.77	0.81	0.00	0.21
	[95% CI]	[0.35, 2.45]	[0.42, 4.55]	[0.67, 0.93]	[-0.46, 0.45]	[-0.18, 0.54]
Panel E: Bekaert, Engstrom, and Ermolov (2020) (with Preference Shocks)						
n_t	**Data-implied**	2.12	1.62	0.81		
	Simulation-implied	2.14	1.19	0.97		
	[95% CI]	[0.38, 6.77]	[0.26, 3.19]	[0.93, 0.99]		
q_t	**Data-implied**	1.00	0.15	0.92	0.54	
	Simulation-implied	1.00	0.15	0.97	0.88	
	[95% CI]	[0.77, 1.51]	[0.04, 0.38]	[0.93, 0.99]	[0.23, 1.00]	
s_t	**Data-implied**	5.4E-3	1.4E-2	0.75	0.83	0.70
	Simulation-implied	3.7E-3	2.8E-3	0.55	0.00	0.08
	[95% CI]	[3.3E-3, 5.1E-3]	[4.3E-5, 1.1E-2]	[0.51, 0.73]	[-0.17, 0.24]	[-0.13, 0.33]
Panel F: Bekaert, Engstrom, and Ermolov (2020) (without Preference Shocks)						
n_t	**Data-implied**	0.08	0.11	0.79		
	Simulation-implied	0.08	0.05	0.97		
	[95% CI]	[0.02, 0.25]	[0.01, 0.14]	[0.93, 0.99]		
q_t	**Data-implied**	1.00	0.18	0.90	0.77	
	Simulation-implied	1.00	0.13	0.97	0.98	
	[95% CI]	[0.87, 1.38]	[0.02, 0.39]	[0.92, 0.99]	[0.85, 1.00]	
Panel G: Gabaix (2012)						
L_t	**Data-implied**	0.70	0.59	0.80		
	Simulation-implied	0.69	0.16	0.97		
	[95% CI]	[0.45, 1.01]	[0.08, 0.29]	[0.93, 0.99]		
Panel H: Wachter (2013)						
λ_t	**Data-implied**	3.0E-3	2.8E-3	0.80		
	Simulation-implied	2.9E-3	1.5E-3	0.97		
	[95% CI]	[6.5E-4, 8.1E-3]	[4.6E-4, 3.6E-3]	[0.93, 0.99]		

Table IA.6

Bollerslev, Tauchen, and Zhou (2009)**-Implied Market Risk Premium Decomposition Summary Statistics**

This table reports summary statistics for the model-implied risk premium decompositions based on Bollerslev, Tauchen, and Zhou (2009) ("BTZ") described in Section IA.7.3.2 with $n = 1$ (i.e., the market risk premium). This table is analogous to Table 6 in the main draft, but just focused on the BTZ model. Panel A reports statistics for the risk premium levels (annualized, in percent) and Panel B reports statistics for the contributions of risk premia from each region to the total risk premium (as fractions of the total risk premium, in percent). A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$ and these labels correspond to the downside, central, and upside risk premia, respectively. $A = A_d \cup A_c \cup A_u$ and this label corresponds to the total risk premium. Statistics reported under "Unconditional" use the full estimated time series for each risk premium measure. Statistics reported under "Cond. Means" report the means for each time series conditional on 30-day risk-neutral variance ($\mathbb{M}^{*(2)}_{t\to T}[A]$) falling below it's first quartile ("Lo"), between its first and third quartiles ("Mid"), or above its third quartile ("Hi"). These correspond to periods of low, moderate, or high market volatility, respectively. Results are based on state variables extracted from the data under each model using their original calibrations, which are monthly in all cases, and use daily data from January, 1996 through June, 2019.

			Panel A: $\mathbb{RP}^{(1)}_{t\to T}[\boldsymbol{A_s}]$ **(%)**					**Panel B:** $\mathbb{RP}^{(1)}_{t\to T}[\boldsymbol{A_s}]/\mathbb{RP}^{(1)}_{t\to T}[\boldsymbol{A}]$ **(%)**				
			Cond. Means			**Unconditional**		**Cond. Means**			**Unconditional**	
Class	**Model**	**Region**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**	**Lo**	**Mid**	**Hi**	**Mean**	**St. Dev.**
LRR	**BTZ**	$\boldsymbol{A}$	-8.52	-4.50	60.38	10.71	144.62					
		$\boldsymbol{A_d}$	-0.02	-26.63	-38.18	-22.87	69.72	0.02	10.28	20.17	10.19	26.50
		$\boldsymbol{A_c}$	-9.46	11.56	24.22	9.47	68.05	99.14	83.05	53.16	79.60	31.33
		$\boldsymbol{A_u}$	0.96	10.58	74.34	24.11	65.91	0.84	6.67	26.67	10.21	21.55

Table IA.7

Average Conditional Differences Between Data- and Model-Implied Decompositions: Bollerslev, Tauchen, and Zhou (2009)

This table reports summary statistics for the conditional differences between the unrestricted data-implied market risk premium decomposition time series (30-day horizon) summarized in Table 2 and the corresponding model-implied market risk premium decomposition time series based on the Bollerslev, Tauchen, and Zhou (2009) model reported in Table IA.6. This table is analogous to Table 8 in the main draft, but just focused on the BTZ model. Panel A reports statistics for the annualized level differences (i.e., $\mathbb{RP}^{(1)}_{t\to T,\,data}\left[A_s\right] - \mathbb{RP}^{(1)}_{t\to T,\,model}\left[A_s\right]$) and Panel B reports statistics for the contribution differences (i.e., $\mathbb{RP}^{(1)}_{t\to T,\,data}\left[A_s\right]/\mathbb{RP}^{(1)}_{t\to T,\,data}\left[A\right] - \mathbb{RP}^{(1)}_{t\to T,\,model}\left[A_s\right]/\mathbb{RP}^{(1)}_{t\to T,\,model}\left[A\right]$). A_d=$[0, 0.9]$, A_c=$[0.9, 1.1]$, and A_u=$[1.1, +\infty)$ and these labels correspond to the downside, central, and upside risk premia, respectively. $A = A_d \cup A_c \cup A_u$ and this label corresponds to the total risk premium. T-statistics are reported in parentheses and are computed according to Newey and West (1987) with lag values according to Newey and West (1994), with one slight modification. Since we have overlapping data, we multiply the Newey and West (1994)-implied lag value by the number of overlapping days (21) and use this as our lag value. Data is daily and runs from January, 1996 through June, 2019.

		Panel A: $\mathbb{RP}^{(1)}_{t\to T}\left[A_s\right]$ differences (%)				Panel B: $\mathbb{RP}^{(1)}_{t\to T}[A_s]/\mathbb{RP}^{(1)}_{t\to T}[A]$ differences (%)		
Class	**Model**	A	A_d	A_c	A_u	A_d	A_c	A_u
LRR	**BTZ**	-2.00	27.32	-6.14	-23.14	35.48	-31.75	-3.73
		(-0.05)	(1.59)	(-0.38)	(-1.66)	(5.49)	(-5.44)	(-0.81)